

Singularities of the Reidemeister torsion form on the character variety

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Abstract

In this paper we aim to study the Reidemeister torsion as a differential form on the character variety $X(M)$ of a compact oriented 3-manifold M with toric boundary. As soon as a component X of $X(M)$ is one dimensional, we define the smooth projective model of a double cover of X , namely the augmented variety Y , and we compute the vanishing order of the Reidemeister torsion at any point of Y . In fact, it occurs at three distinct kind of points of Y : singular finite points, intersection points with the "component of reducible type", and "ideal" points. We show that in the first case, the vanishing order is related with the type of singularity we encounter, in the second case we relate it with the Alexander module of M , and in the third case we bound the vanishing order by the Euler characteristic of an incompressible surface associated to the ideal point by the Culler-Shalen theory.

0 Introduction

The Reidemeister torsion is a combinatorial topological invariant, celebrated in the 30's for being able to distinguish non homeomorphic lens spaces, and to complete their classification. Specifically, the Reidemeister torsion is a topological invariant $\text{tor}(M, \text{Ad}_\rho)$ where M is a 3-dimensional manifold and ρ is a representation of $\pi_1(M)$ into a Lie group G . If the twisted cohomology groups $H^*(M, \text{Ad}_\rho)$ vanish, the torsion is a numerical invariant defined up to sign. In the general case, we may interpret it as a volume element in the twisted cohomology, that is an element in

$$\text{Det}(H^*(M, \text{Ad}_\rho)) = \bigotimes_{i=0}^3 \text{Det}(H^i(M, \text{Ad}_\rho))^{(-1)^i}.$$

Moreover if ρ and ρ' are conjugated representations, there is a natural isomorphism $\text{Det}(H^*(M, \text{Ad}_\rho)) \simeq \text{Det}(H^*(M, \text{Ad}_{\rho'}))$ that preserves the torsion. Hence it is natural to define the Reidemeister torsion as a section of some line bundle over the character variety.

When M is a 3-manifold with toric boundary (e.g. a knot complement), and $G = \text{SL}_2(\mathbb{C})$, Joan Porti in his Phd thesis [Por97] defined the torsion as an analytic function on a Zariski open subset of the character variety depending on a choice of a boundary curve. Many computations have been performed by J. Dubois and al. [Dub06], [DHY09] and the torsion has been extended to the whole character variety in [DG09]. We will follow in this article the approach of [Mar15], where the Reidemeister torsion of any 3-manifold with boundary is interpreted as a rational volume form on the character variety. More precisely, if the

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boundary of M is a torus, the torsion is a rational volume form on the *augmented character variety* which is the following 2-fold covering of the character variety:

$$\bar{X}(M) = \{(\rho : \pi_1(M) \rightarrow \mathrm{SL}_2(\mathbb{C}), \lambda : \pi_1(\partial M) \rightarrow \mathbb{C}^*), \mathrm{Tr} \rho|_{\pi_1(\partial M)} = \lambda + \lambda^{-1}\} // \mathrm{SL}_2(\mathbb{C}).$$

In this paper we assume that $X(M)$ is 1-dimensional and *reduced*, in the sense of schemes. The first assumption is guaranteed by the assumption that M is *small*, that is without closed incompressible surfaces.

Let X be an irreducible component of $\bar{X}(M)$ containing the character of an irreducible representation and let Y be its smooth projective model. It is a smooth compact curve obtained from Y by desingularizing and adding a finite number of points "at infinity": we call the latter points "ideal" points of Y and the others are "finite" points. We will denote by v an element of Y that we may view as a valuation on the function field $\mathbb{C}(Y) = \mathbb{C}(X)$. Its local ring at v will be denoted by \mathcal{O}_v . The torsion will be denoted by $\mathrm{tor}(M)$ and seen as an element of $\Omega_{\mathbb{C}(Y)/\mathbb{C}}$. The first result in this article is the following theorem :

Theorem 0.1. *Let v be a finite point of Y .*

1. *If v projects to an irreducible character in $X(M)$, then the vanishing order of $\mathrm{tor}(M)$ at v is the length of the torsion part of the module $\Omega_{\mathbb{C}[X]/\mathbb{C}} \otimes \mathcal{O}_v$. This integer is an invariant of the local singularity which can be computed explicitly. In particular if v projects to a smooth point of X then the torsion is regular at v .*
2. *Suppose that M is a knot complement and m is a meridian. If v projects to a reducible character $\lambda + \lambda^{-1}$ in $X(M)$ then $\lambda(m)^2$ is a root of the Alexander polynomial of M of order $r \geq 1$. Under some technical hypothesis detailed in Section 3, $\mathrm{tor}(M)$ vanishes at v at order $2r - 2$.*

If v is an ideal point of Y , then the Culler-Shalen theory associates to v an action of $\pi_1(M)$ on the Bass-Serre tree of $\mathrm{SL}_2(\mathcal{O}_v)$ which itself produces an incompressible surface Σ in M . We say that Σ is associated to the ideal point v .

Theorem 0.2. *Let v be an ideal point of Y and Σ be an incompressible surface associated to v . We suppose that Σ is a connected surface splitting M into two handlebodies and that Y contains the character of a representation whose restriction to Σ is irreducible. Then the torsion $\mathrm{tor}(M)$ has vanishing order at v bounded by $-\chi(\Sigma) - 1$.*

We will say that a surface $S \subset M$ is *free* if its complement is a union of handlebodies. Many natural constructions yield such surfaces, for example take a knot diagram and consider the checkerboard surfaces (on Figure 1 on the left is an example of such a surface). If one, say Σ , is an incompressible non orientable surface in M , then the boundary of a neighborhood of Σ is orientable, and does split M into two handlebodies, as can be easily seen (both part of its complement retract onto a graph).

We deduce from this theorem an unexpected relation between the genus of the character variety of M and the genus of the incompressible surfaces in M . More precisely, suppose that M is a knot complement whose character variety is one dimensional. Pick a smooth component of the variety, and assume that each ideal point $y \in Y$ corresponds to an incompressible surface Σ_y that verifies the hypothesis of the theorem. Furthermore assume that the Alexander polynomial of M has only simple roots. Then

$$-\chi(Y) \leq \sum_y (-\chi(\Sigma_y) - 1).$$

Example 0.3. We know from [HT85] that the knot 5.2 has two incompressible surfaces in its complement : Σ_1 whose Euler characteristic is -4 , and Σ_2 whose Euler characteristic is -2 (see figure 1).

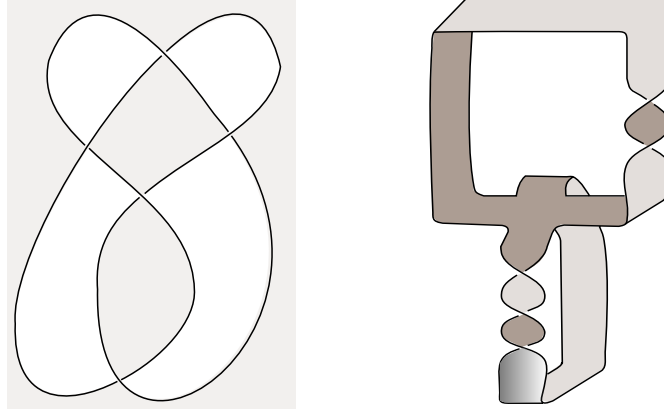


Figure 1: Incompressible surfaces in the complement of the knot 5.2. The surface Σ_1 is the orientation covering of the non oriented surface here colored on the left, that can be thought as the boundary of a tubular neighborhood of this non orientable surface. The surface Σ_2 is again the orientation covering of the surface colored on the right, that can be obtained as follows : consider two parallel copies of each twisted bands above and below the square in the middle, and plumb them along this square. The result is connected because the bands below have an odd number of twists, and this is our surface Σ_2

The (geometric component of the) character variety has 3 ideal points, two of them corresponding to Σ_2 , and the third to Σ_1 . The torsion vanishes at order 1 on the Σ_2 's ideal points, and at order 3 at the other. Hence on the augmented variety Y , one obtains $-\chi(Y) = 4 \times 1 + 2 \times 3 = 10$. The covering map $Y \rightarrow X(M)$ ramifies on six points, hence $-\chi(X) = \frac{1}{2}(-\chi(Y) - 6) = 2$, and the genus of X is 2, as it can be verified directly.

Question 0.4. Is the inequality of Theorem 0.2 an equality?

It is the case of all examples we have listed in Section 2.2. On the other hand, a careful examination of the proof shows that it has to be generically the case, the lack of equality should be interpreted as a non-transversal situation.

Pursuing the last remark, it seems reasonable to think that the vanishing order of the torsion is always positive. In general one may ask the following question :

Question 0.5. For which knot complements is the torsion a regular differential form?

It is the case from Theorem 0.1 on the affine part of $\bar{X}(M)$, but the torsion could have a pole at infinity. Actually, the examples of torus knots (p, q) provide a case when it does : each (non-abelian) component of the character variety is isomorphic to \mathbb{P}^1 with an unique ideal point, corresponding to an essential annulus in the complement. Although Theorem

0.2 cannot apply because fundamental groups of annuli have no irreducible representations, one can compute directly that the torsion has a pole of order one at those points.

Further work should concern the study of reducible characters that are limits of irreducible characters, as in Theorem 0.1, item 2 : to all appearances it is possible to weaken the hypothesis of this theorem. But it would be needed to have a better understanding of the relation between the first twisted cohomology group and the tangent space at this point ; which is a hard problem, partially answered in [FK91, HPSP01].

An other natural continuation is the study of other situations in Theorem 0.2. Probably, the same techniques may apply to the non-separating case $M \setminus \Sigma$ connected, with Σ free again. The case where Σ is not free makes the approach significantly more complicated.

The case when the ideal point is a reducible character of $X(\Sigma)$ could be explored too, but it is highly non generic. We only know the case of torus knots for which the computation can be done directly.

Those remarks leads us to the following questions :

Question 0.6. Are there (hyperbolic) knots in \mathbb{S}^3 whose character variety holds a genus 0 component that contains the character of an irreducible representation ?

As there is no regular differential forms on genus 0 curves, it would contradict one of the previous statements. On the other hand, progress on this question should help understanding which knot's groups surjects on a torus knot's group. Such a surjection would induce a reversed injection at the level of character varieties, and thus an embedding of genus 0 components in the character variety of the knot. Then either the variety is more than two-dimensional, either it contains spherical components. Notice that all known example of knots whose fundamental group surjects on the trefoil's are at least 3-bridge knots, hence there is no evidence for their character variety to be one-dimensional. An explicit computation should be interesting, but seems technically difficult.

Question 0.7. Can we compute explicetely the character variety of a knot whose group surjects on the trefoil's, and observe if it holds a one-dimensional component of genus 0?

The paper is organized as follows : in Section 1 we give several definitions about character varieties, tautological representations, and we treat some examples. In Section 2 we define the Reidemeister torsion of a complex, and explain why in our case it can be seen as a differential form on a double covering of the character variety that we have constructed in Section 1.2, and compute it on many examples. In Section 3 we treat the "finite case" : we give the statement, the proof and the interpretation of the Main Theorem 1, that corresponds to Theorem 0.1 here ; and in Section 4 we treat the "ideal case" : we state and prove the Main Theorem 2 that corresponds to Theorem 0.2 here. We end by checking our results on all examples treated in Section 2.2, and making some observations that motivate the former questions and remarks.

Throughout this paper, k will be an algebraically closed field of characteristic 0, M will be a 3 dimensional, compact, connected manifold with boundary a torus and rational homology of a circle (e.g. a knot complement), and we will denote by $\Gamma = \pi_1(M)$ its fundamental group.

1 Character variety and tautological representation

1.1 The character variety

We give two equivalent definitions of the character variety :

Definition 1.1. Let $A[\Gamma]$ be the algebra $A[\Gamma] = k[X_{ij}^\gamma, 1 \leq i, j \leq 2, \gamma \in \Gamma] / (\det(X^\gamma) - 1, X^{\gamma\delta} - X^\gamma X^\delta)$, where X^γ denotes the matrix (X_{ij}^γ) . We define the *representation variety* $R(\Gamma) = \text{Hom}(\Gamma, \text{SL}_2)$ of the group Γ (or of M) as the spectrum of this algebra. More concretely, given a set of n generators of Γ , it is the affine sub-variety of k^{4n} defined by the polynomial equations given by relations in Γ .

The group $\text{SL}_2(k)$ acts on $R(\Gamma)$ by conjugation, hence we define the *character variety* of M by the algebro-geometric quotient :

$$X(M) = R(M) // \text{SL}_2(k)$$

By definition, it is the spectrum of the algebra of invariants $A[\Gamma]^{\text{SL}_2}$.

Remark 1.2. In Example 1.7 we give an illustration of such a quotient. The main difference with the usual one is that it identifies two orbits in $R(M)$ iff their closure intersect.

Remark 1.3. In general, the use of the term "variety" is reserved to irreducible and reduced algebraic sets. An *irreducible* set is a set which is not a reunion of two proper closed subsets. Given a ring R , $\text{Spec}(R)$ is said to be *reduced* if R contains no nilpotent elements. We will call $X(M)$ a variety despite it has no reason to be irreducible, nor reduced.

Definition 1.4. Let $B[\Gamma]$ be the algebra $B[\Gamma] = k[Y_\gamma, \gamma \in \Gamma] / (Y_1 - 2, Y_{\gamma\delta} + Y_{\gamma\delta^{-1}} - Y_\gamma Y_\delta)$. One can prove ([CS83, Prop. 1.4.1]) that it is finitely generated. Define $X'(M) = \text{Spec } B[\Gamma]$.

Proposition 1.5 ([Pro87, PS00]). *There is an isomorphism $X(M) \simeq X'(M)$.*

It furnishes an alternative definition of the character variety.

Remark 1.6. The relations of the algebra $B[\Gamma]$ correspond to the well-known trace relation $\text{Tr } A \text{ Tr } B = \text{Tr } AB + \text{Tr } AB^{-1}$ in $\text{SL}_2(k)$. Hence, we will often use the terminology *trace functions* when speaking of the Y_γ 's.

This definition of the character variety provides a relation with the *Kauffman skein module* : consider $\mathcal{S}_A(M)$ the $k[A^{\pm 1}]$ -module generated by isotopy classes of link in M , quotiented by the so-called Kauffman relations. Then the specialization $S_{-1}(M) = \mathcal{S}_A(M) \otimes_{A=-1} k$ is isomorphic to $B[\Gamma]$ via the morphism $[\gamma] \mapsto Y_\gamma$.

1.2 The augmented variety

In order to motivate the following definitions, we begin this section by giving an example.

Example 1.7 (The character variety $X(\partial M)$). Let $\rho : \mathbb{Z}^2 \rightarrow \text{SL}_2(k)$ be a morphism. For any $\gamma \in \pi_1(\partial M)$, the trace function $Y_\gamma(\rho) = \lambda(\gamma) + \lambda^{-1}(\gamma)$ with $\lambda(\gamma), \lambda^{-1}(\gamma)$ the eigenvalues of $\rho(\gamma)$, notice that $\lambda \in H^1(\partial M, k^*)$. Moreover, the involution σ of $H^1(\partial M, k^*)$ that map λ on λ^{-1} fixes every Y_γ , hence one obtains the isomorphism $X(\partial M) \simeq H^1(\partial M, k^*)^\sigma$

that maps $[\rho]$ on $\lambda + \lambda^{-1}$. This morphism is well defined because every $\rho : \pi_1(\partial M) \rightarrow \mathrm{SL}_2(k)$ of the form $\begin{pmatrix} \lambda & * \\ 0 & \lambda^{-1} \end{pmatrix}$ up to conjugacy, and is surjective. The important fact we have to mention here is that even if $\lambda(\gamma) = \pm 1$, then in the algebraic quotient $\rho(\gamma) = \begin{pmatrix} \pm 1 & * \\ 0 & \pm 1 \end{pmatrix}$ is identified with the matrix $\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$; it proves the injectivity.

On the other hand, the variety $H^1(\partial M, k^*)$ is the *eigenvalue variety* of ∂M : a point of the latter is a choice of a specific eigenvalue for a class of representations $[\rho]$. Denote by $\bar{B}[\pi_1(\partial M)] = k[Z_\gamma, \gamma \in \pi_1(\partial M)] / (Z_\gamma Z_\delta - Z_{\gamma\delta})$ its function ring, where functions are of the form $Z_\gamma(\lambda) = \lambda(\gamma)$. One has a natural map $B[\pi_1(\partial M)] \rightarrow \bar{B}[\pi_1(\partial M)]$ that sends Y_γ onto $Z_\gamma + Z_\gamma^{-1}$.

Definition 1.8. We define the *augmented representation variety* $\bar{R}(M)$ to be the subvariety of $R(M) \times H^1(\partial M, k^*)$ given by the pairs $\{(\rho, \lambda), \rho : \pi_1(M) \rightarrow \mathrm{SL}_2(k), \lambda : \pi_1(\partial M) \rightarrow k^*, \lambda(\gamma) + \lambda(\gamma)^{-1} = \mathrm{Tr} \rho(\gamma) \ \forall \gamma \in \pi_1(\partial M)\}$.

Again $\mathrm{SL}_2(k)$ acts on $\bar{R}(M)$, trivially on the second factor, thus we define the *augmented character variety* to be the quotient $\bar{X}(M) = \bar{R}(M) // \mathrm{SL}_2(k)$.

The advantage of this two-folds covering is the following : on one hand the functions of $X(M)$ are trace functions, on the other hand on $\bar{X}(M)$ we have at disposal, for any $\gamma \in \pi_1(\partial M)$, two *eigenvalue functions* $Z_{\gamma \pm 1}$ that maps a class of pair (ρ, λ) on the choice of an eigenvalue $\lambda(\gamma)$ for $\rho(\gamma)$.

Definition 1.9. Denote by $\bar{B}[\Gamma] = B[\Gamma] \otimes_{B[\pi_1(\partial M)]} \bar{B}[\pi_1(\partial M)]$ and we define the augmented character variety as the fibered product :

$$\bar{X}(M) = X(M) \times_{X(\partial M)} H^1(\partial M, k^*)$$

that is, $\bar{X}(M) = \mathrm{Spec} \bar{B}[\Gamma]$.

1.3 Characters

Definition 1.10. A *character* is defined to be a point of the character variety. Without more precision, we will mean a k -point, that is a morphism $B[\Gamma] \rightarrow k$, but k should be replaced by any k -algebra R . Any representation $\rho : \Gamma \rightarrow \mathrm{SL}_2(R)$ induces the following character $\chi_\rho(Y_\gamma) = \mathrm{Tr}(\rho(\gamma))$.

We skip the proof of the following lemma.

Lemma 1.11. *Given a finitely generated group Γ , and any field K , a representation is absolutely irreducible (that is irreducible in \bar{K}) iff there exists $\alpha, \beta \in \Gamma$ such that $\mathrm{Tr} \rho[\alpha, \beta] \neq 2$.*

Denote by $\Delta_{a,b} = Y_\alpha^2 + Y_\beta^2 + Y_{\alpha\beta}^2 - Y_\alpha Y_\beta Y_{\alpha\beta} - 4 = Y_{[\alpha,\beta]} - 2 \in B[\Gamma]$. This suggests us the following definition :

Definition 1.12. An R -character χ will be said *irreducible* if there exists $\alpha, \beta \in \pi_1(M)$ such that $\chi(\Delta_{\alpha,\beta}) \in R^\times$. If not, χ is said *reducible*. It is a Zariski closed condition, hence we will say that a component that contains only reducible character is a *component of reducible type*. On the other hand, a component that contains at least one irreducible character will be said a *component of irreducible type*. Finally, χ is *central* if $\chi(Y_\gamma)^2 = 4$ for all $\gamma \in \Gamma$.

The following theorem allows us to identify irreducible characters with (conjugacy class of) representations :

Theorem 1.13 ([Sai94, Mar15]). *If the Brauer group of K is trivial (for instance if K is algebraically closed, or the function field of a curve over an algebraically closed field), then to any irreducible K -character χ one can associate a representation $\rho : \Gamma \rightarrow \mathrm{SL}_2(K)$ whose character is χ . Moreover, it is unique up to conjugacy by an element of $\mathrm{GL}_2(K)$. It implies that on the irreducible part of the character variety, the algebraic quotient is in fact a topological quotient.*

Definition 1.14. The *fiber* of an R -character χ is the set of representations $\rho : \Gamma \rightarrow \mathrm{SL}_2(R)$ that map onto χ by the algebraic quotient map.

Remark 1.15. The theorem says that in the fiber of an irreducible character, there are only conjugated representations, hence it is 3-dimensional.

This is not true in the reducible case, as was illustrated in Example 1.7. More precisely, if the fiber of a reducible character contains only abelian representations (i.e. such that the image of Γ is an abelian group), then all of them are still conjugated and one says that this character is *abelian*. If not, we will call it *metabelian*, and its fiber contains both abelian and reducible non abelian representations.

More precisions about those facts can be found in [Por97, Section 3.2].

1.4 The tautological representation

If Y is a component of $\bar{X}(M)$, we will call it of irreducible type if it projects on a component of irreducible type of $X(M)$. In the sequel we will assume that $\bar{X}(M)$ is reduced. Notice that there are no reasons why character varieties of 3 manifolds with toric boundary should be reduced although no example of not reduced such varieties are known.

Pick Y a component of $\bar{X}(M)$ of irreducible type, it corresponds to a minimal prime ideal \mathfrak{p} of $\bar{B}[\Gamma]$ such that $k[Y] = \bar{B}[\Gamma]/\mathfrak{p}$ is the function algebra of Y , and the morphism $\bar{B}[\Gamma] \rightarrow k[Y]$, seen as a $k[Y]$ -character, is irreducible. Denote by $k(Y)$ the fraction field of $k[Y]$, and by χ_Y the composition $\bar{B}[\Gamma] \rightarrow k[Y] \rightarrow k(Y)$. The following is an immediate consequence of Theorem 1.13 :

Proposition 1.16. *Assume that Y is one dimensional. Then there is a representation $\rho_Y : \Gamma \rightarrow \mathrm{SL}_2(k(Y))$, called the tautological representation, defined up to conjugacy, whose character is χ_Y .*

In some sense, this representation is a family of representations parametrized by the points of Y , as will emphasize the following examples.

Example 1.17 ($\Gamma = \mathbb{Z}$, the reducible component). We prove here that $X(\mathbb{Z}) \simeq k$: every representation $\rho : \mathbb{Z} \rightarrow \mathrm{SL}_2(k)$ is equivalent to $\rho(n) = \begin{pmatrix} t^n & 0 \\ 0 & t^{-n} \end{pmatrix}$, for some $t \in k^*$. Seen as a representation in $\mathrm{SL}_2(k(t))$, it is the tautological representation of a double covering of $X(\mathbb{Z})$ (one can think about it as $\bar{X}(\mathbb{Z})$) : the function field of $X(\mathbb{Z})$ is $k(u)$, with $t+t^{-1} = u$. If $H_1(M) \simeq \mathbb{Z}$, any abelian representation factorizes through \mathbb{Z} , hence there is an unique reducible component in $X(M)$, and it is isomorphic to k .

Example 1.18 ($\Gamma = \langle \alpha, \beta \rangle$). Here the algebra $B[\Gamma]$ is generated by $Y_\alpha, Y_\beta, Y_{\alpha\beta}$. Moreover, those form a basis : given a relation $P(Y_\alpha, Y_\beta, Y_{\alpha\beta}) = 0$, one proves $P = 0$ setting $\rho(\alpha) = \begin{pmatrix} X & 1 \\ -1 & 0 \end{pmatrix}$ and $\rho(\beta) = \begin{pmatrix} 0 & -Z^{-1} \\ Z & Y \end{pmatrix}$. One gets $P(X, YZ + Z^{-1}) = 0$, hence P is trivial in $k(X, Y, Z)$. The trace ring is then $k[X, Y, Z + Z^{-1}]$, and the character variety is isomorphic to k^3 . The representation ρ defined below is the tautological representation in $\mathrm{SL}_2(k(X, Y, Z))$ and there is no reason that it exists in $\mathrm{SL}_2(k(X, Y, Z + Z^{-1}))$, because the Brauer group of this field is not trivial.

Example 1.19 (The trefoil knot). Here M is the complement of the trefoil knot in \mathbb{S}^3 , $\pi_1(M) = \langle a, b \mid a^2 = b^3 \rangle$. Denote by $z = a^2 = b^3$, it generates the center of Γ . Hence any irreducible representation ρ needs to map z onto $\pm \mathrm{Id}$. If $\rho(z) = \mathrm{Id}$, then $\rho(a) = -\mathrm{Id}$ and necessarily ρ becomes abelian, thus we fix $\rho(z) = -\mathrm{Id}$. Up to conjugacy, fix $\rho(b) = \begin{pmatrix} -j & 0 \\ 0 & -j^2 \end{pmatrix}$. One can still conjugate ρ by diagonal matrices without change on $\rho(b)$, thus one can fix the right-upper entry of $\rho(a)$ to be equal to 1 ; and as $\rho(a)^2 = -\mathrm{Id}$, the Cayley-Hamilton theorem implies that $\mathrm{Tr} \rho(a) = 0$, hence $\rho(a) = \begin{pmatrix} t & 1 \\ -(t^2 + 1) & -t \end{pmatrix}$, for some $t \in k$. As $(j - j^2)t = \mathrm{Tr}(ab^{-1})$, the function field of the component of irreducible type X is $k(t)$; and $X \simeq k$. The latter ρ is the tautological representation. The augmented character variety is obtained by picking an eigenvalue of any boundary curve. Consider the meridian ab^{-1} , its trace is $(j - j^2)t$, hence the field extension $u + u^{-1} = (j - j^2)t$ provides a double covering $Y \rightarrow X$, that ramifies twice, when $(j - j^2)^2 t^2 = 4$. Notice that Y has again genus 0.

too.

Example 1.20 (The figure-eight knot). Here M denotes the complement of the figure-eight knot in \mathbb{S}^3 , $\pi_1(M) = \langle u, v \mid vw = wu \rangle$ with $w = [u, v^{-1}]$. Notice that the meridians u and v are conjugated, hence their trace function are equal. Denote by $x = Y_u = Y_v$, and by $y = Y_{uv}$, then $B[\pi_1(M)] = k[x, y]/(P)$ where $P(x, y) = (x^2 - y - 2)(2x^2 + y^2 - x^2y - y - 1)$ is obtained by expanding the relation $\mathrm{Tr} vwu^{-1}w^{-1} = 2$. The first factor of P is the equation of the reducible component, denote by X the curve defined by the second factor of P , it is a smooth plane curve of genus 1. The augmented variety $Y \rightarrow X$ is described as follows : add the equation $\alpha + \alpha^{-1} = x$, it ramifies at four points $\{x^2 = 4, y^2 - 5y + 7 = 0\}$, hence Y has genus 3.

The tautological representation $\rho : \Gamma \rightarrow k(Y)$ can be defined as follows :

$$\rho(u) = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha^{-1} \end{pmatrix}, \rho(v) = \begin{pmatrix} \alpha & 0 \\ y - \alpha^2 - \alpha^{-2} & \alpha^{-1} \end{pmatrix}$$

1.5 Cotangent space and twisted homology

In the sequel, by twisted (co)-homology group we will mean the (co)-homology of M with coefficient in $\mathrm{sl}_2(k(Y))$ "twisted" by the $\pi_1(M)$ -action through $\mathrm{Ad} \circ \rho$. We will denote it by $H_*(M, \mathrm{Ad}_\rho)$.

It is well-known from Weil (see for instance [LM85, Wei64]) that the Zariski tangent-space at an irreducible character is encoded by the first twisted-cohomology group as soon as the variety is reduced at this character. Here we give a global counterpart of this statement.

Definition 1.21 (Kähler differentials). Let A be a ring, and B an A -algebra, the B -module of A -derivations $\Omega_{B/A}$ is the free B -module generated by the symbols $db, b \in B$ quotiented by the ideal generated by relations $\{da = 0, d(b_1 + b_2) = db_1 + db_2, d(b_1 b_2) = b_1 db_2 + b_2 db_1\}$. Given an affine variety X on a field k , we will denote by $\Omega_{X/k}$ the ring of Kähler differentials $\Omega_{k[X]/k}$. The set of rational Kähler differentials is the $k(X)$ vector-space $\Omega_{k(X)/k}$, it is isomorphic to $\Omega_{k[X]/k} \otimes k(X)$.

Proposition 1.22 (see Prop 4.1 of [Mar15]). *Let Y be a component of irreducible type of $\bar{X}(M)$, $k[Y] = \bar{B}[\Gamma]/\mathfrak{p}$ its function ring and $\rho : \Gamma \rightarrow \mathrm{SL}_2(k(Y))$ the tautological representation. There are isomorphisms*

$$H_1(M, \mathrm{Ad}_\rho) \simeq \Omega_{\bar{X}(M)/k} \otimes_{\bar{B}(\Gamma)} k(Y) \simeq \Omega_{k(Y)/k}$$

Sketch of proof. For any $\gamma \in \Gamma$, denote by $\rho(\gamma)_0$ the trace-free matrix $\rho(\gamma) - \frac{\mathrm{Tr} \rho(\gamma)}{2} \mathrm{Id}$. The morphism of $\bar{B}[\Gamma]$ -modules

$$\begin{aligned} \Omega_{\bar{B}[\Gamma]/k} &\rightarrow H_1(M, \mathrm{Ad}_\rho) \\ dY_\gamma \otimes 1 &\mapsto \rho_0(\gamma) \otimes [\gamma] \end{aligned}$$

extends to a $k(Y)$ -linear map $\Omega_{\bar{B}[\Gamma]/k} \otimes k(Y) \rightarrow H_1(M, \mathrm{Ad}_\rho)$. An inverse map can be produced with the help of Saito's Theorem ([Sai94]), which proves the first isomorphism. The second is a consequence of the fact that $\mathfrak{p}/\mathfrak{p}^2 = 0$. \square

Again Y is a component of $\bar{X}(M)$ of irreducible type, reduced, and assume it is one-dimensional. Notice that the last property follows from the assumption that M is *small* (i.e. does not contain any closed incompressible surface), or that Y projects onto a *geometric component* of $X(M)$ (that carries the character of a representation corresponding to a hyperbolic structure).

Proposition 1.23. *Denote by H the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and by $r^* : H^i(M, \mathrm{Ad}_\rho) \rightarrow H^i(\partial M, \mathrm{Ad}_\rho)$ the morphism induced by the inclusion $\partial M \subset M$. Then $H^0(M, \mathrm{Ad}_\rho) = H^i(M, \mathrm{Ad}_\rho) = 0$ for $i \geq 3$; $H^1(M, \mathrm{Ad}_\rho)$ is one dimensional and $H^2(M, \mathrm{Ad}_\rho) \simeq k(Y)$, via by $\eta \mapsto \mathrm{Tr}(r^* \eta(\cdot) H)$.*

Proof. As $\partial M \neq \emptyset$, M admits a cellular decomposition with only 0, 1 and 2-cells, hence $H^i(M) = 0, \forall i \geq 3$. By definition $H^0(M, \mathrm{Ad}_\rho)$ is the set of Ad_ρ -invariants vectors, hence it is trivial because ρ is not abelian. A classical equality is that $\dim \Omega_{k(Y)/k} = \dim Y$, thus Proposition 1.22 together with Universal Coefficients Theorem imply that $\dim H^1(M, \mathrm{Ad}_\rho) = 1$. The Euler characteristic of M is 0, thus $\dim H^2(M, \mathrm{Ad}_\rho) = 1$ too. Let's explicit the last isomorphism : the long exact sequence of the pair $(M, \partial M)$ ends with

$$\dots \rightarrow H^2(M, \mathrm{Ad}_\rho) \xrightarrow{r^*} H^2(\partial M, \mathrm{Ad}_\rho) \rightarrow H^3(M, \mathrm{Ad}_\rho)$$

Poincare duality makes the last term vanish. As $\pi_1(\partial M)$ is abelian, $H^0(\partial M, \mathrm{Ad}_\rho)$ is not trivial, and so is $H^2(\partial M, \mathrm{Ad}_\rho)$; hence r^* is an isomorphism.

Now we use the construction of the augmented variety : up to conjugacy the restriction $\rho : \pi_1(\partial M) \rightarrow \mathrm{SL}_2(k(Y))$ is of the form $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, hence its adjoint action on $\mathfrak{sl}_2(k(Y))$ leaves the vector space spanned by H invariant. In other words, $H^0(\partial M, \mathrm{Ad}_\rho)$ is generated by H , and the result follows. \square

1.6 Projective model and action on trees

Definition 1.24. Given a curve X on an algebraically closed field of characteristic 0, there is a canonical way to produce a smooth projective curve \hat{X} birational to X , called the *smooth projective model*. One way is the following : consider \hat{X} to be the set of *discrete valuations* on $k(X)$ endowed with the cofinite topology. To a valuation on $k(X)$ correspond a point (eventually at infinity) of X , and reciprocally to any smooth point $x \in X$ one associates the valuation $v_x : k(X)^* \rightarrow \mathbb{Z}$ whose value at P is the vanishing order of P at x . Define the *local ring* (or valuation ring) of $x \in \hat{X}$ to be $\mathcal{O}_v = \{P \in k(X), v(P) \geq 0\}$. Pick $t \in \mathcal{O}_v$ an element of valuation 1.

There is a birational map $\nu : \hat{X} \rightarrow X$, $x \in \hat{X}$ is an *ideal point* if ν is not defined at x , and the other points are *finite points*.

In the sequel of the paper, Y will be the smooth projective model of a component of irreducible type of $\bar{X}(M)$.

To any point v of Y , the Culler-Shalen theory associates an action of $\pi_1(M)$ on a simplicial tree T_v through ρ . We won't recall the details of this construction, and we refer to [Sha02, Section 3] for precisions, we just say that the vertices of T_v are homothety classes of \mathcal{O}_v -lattices in $k(Y)^2$, and that the action of ρ is induced by the natural action of $\mathrm{SL}_2(k(Y))$ on $k(Y)^2$.

For now, we need the following description of stabilizers of vertices :

Lemma 1.25. *A representation ρ stabilizes a vertex of T_v iff its image is conjugated to a subgroup of $\mathrm{SL}_2(\mathcal{O}_v)$.*

Hence we have the following proposition :

Proposition 1.26. *A point $v \in Y$ is a finite point iff the tautological representation takes value in $\mathrm{SL}_2(\mathcal{O}_v)$, up to conjugacy in $\mathrm{GL}_2(k(Y))$.*

Proof. By definition v is finite iff $v(Y_\gamma) \geq 0$ for all $\gamma \in \Gamma$, that is $\rho(\gamma)$ is conjugated to a matrix in $\mathrm{SL}_2(\mathcal{O}_v)$. Now, thanks to Lemma 1.25, we know that it is equivalent to say that for all $\gamma \in \Gamma$, $\rho(\gamma)$ fixes a vertex in T_v . The following lemma permits us to conclude that the whole group Γ fixes a vertex, and thus, up to conjugacy, $\rho(\Gamma) \subset \mathrm{SL}_2(\mathcal{O}_v)$. \square

Lemma 1.27 (See Corollaire 3, p.90 of [SB77]). *If each element $\gamma \in \Gamma$ fixes a vertex of T_v , then Γ fixes a vertex of T_v .*

Definition 1.28. We will say a representation $\rho : \pi_1(M) \rightarrow \mathrm{SL}_2(k(Y))$ is *convergent* at $v \in Y$ if it takes values in $\mathrm{SL}_2(\mathcal{O}_v)$. In the upcoming sections, we will consider (co)-homology groups with coefficients in $\mathrm{sl}_2(\mathcal{O}_v)$, thanks to Proposition 1.26. We will denote those \mathcal{O}_v -modules by $H^*(M, \mathrm{Ad}_\rho)_v$.

Definition 1.29. Given a convergent representation ρ , the evaluation map $\mathcal{O}_v \rightarrow k$ induces a *residual representation* $\bar{\rho} : \Gamma \rightarrow \mathrm{SL}_2(k)$, and we will study residual cohomology groups $H^*(M, \mathrm{Ad}_{\bar{\rho}})$ too. The tautological representation will be said *residually reducible (resp irreducible, abelian, central)* at a finite point v if the corresponding residual representation $\bar{\rho}$ is reducible (resp irreducible, abelian, central).

1.7 Twisted cohomology with $\mathrm{sl}_2(\mathcal{O}_v)$ coefficients

In this section we collect some lemmas that will be useful later on this paper. Let v a finite point of Y , and $\rho : \Gamma \rightarrow \mathrm{SL}_2(\mathcal{O}_v)$ the tautological representation. We prove the following lemma :

Lemma 1.30. *If ρ is not residually central, there exists $g \in \mathrm{GL}_2(k(Y))$ such that $g\rho g^{-1} : \Gamma \rightarrow \mathrm{SL}_2(\mathcal{O}_v)$ is not residually abelian.*

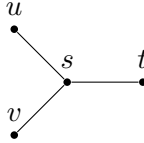
Remark 1.31. Notice that we allow conjugacy by elements of $\mathrm{GL}_2(k(Y))$. Although the complex $C^*(M, \mathrm{Ad}_\rho)_v$ depends on a choice of ρ , the torsion does not, because it is an invariant of the complex of $k(Y)$ vector-spaces $C^*(M, \mathrm{Ad}_\rho)$.

Proof. The proof uses tree-theoretical arguments, we refer to [SB77] for an expanded exposition on the topic.

Let T_v be the Bass-Serre tree for the valuation v , and denote by $T' \subset T$ the subset of fixed points under the action of ρ . By Lemma 1.25, $T' \neq \emptyset$, and we will prove that it is a segment.

T' is of finite length, because if not it would contain an half-line. In [SB77, Chapter 2, Section 1.3] is the following description of stabilizers of half-lines : they fix a line in the completed plane $\hat{\mathcal{O}}_v^2$, in particular in this case ρ would be reducible.

If T' contains a vertex s of valence at least 3 as follows :



then the edges st, su and sv do represent 3 distinct fixed-points of $\mathbb{P}^1(k)$ under the action of $\bar{\rho}$ (again [SB77, Chapter 2, Section 1.3]). But an automorphism of $\mathbb{P}^1(k)$ that fixes 3 points is $\pm \mathrm{Id}$, hence $\bar{\rho}$ is central.

Thus we have proved that T' is of the following form :



Fix an appropriate basis such that each s_i does represent the lattice $\mathcal{O}_v \oplus (t^i)\mathcal{O}_v$. As ρ fixes s_i , one can write for all $\gamma \in \Gamma$, $\rho(\gamma) = \begin{pmatrix} a(\gamma) & b(\gamma) \\ c(\gamma) & d(\gamma) \end{pmatrix}$, with $c(\gamma) \in (t^i)$. On the other

hand, if $b(\gamma) \in (t)$ for all $\gamma \in \Gamma$, then the conjugation by the matrix $U = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}$ acts as a translation on the right on T' , and $U\rho U^{-1}$ remains in $\mathrm{SL}_2(\mathcal{O}_v)$. But $\mathrm{SL}_2(\mathcal{O}_v)$ fixes s_0 by definition, hence $U\rho U^{-1}.s_0 = s_0$ and $\rho.U^{-1}s_0U = U^{-1}s_0U$, but $U^{-1}s_0U \notin T'$, a contradiction.

We conclude that there exists $\gamma \in \Gamma$ such that $b(\gamma) \in \mathcal{O}_v^\times$, and that $\bar{\rho}$ is not abelian. \square

Now we describe the homology groups $H_i(M, \mathrm{Ad}_\rho)_v$.

Lemma 1.32. *If the tautological representation is not residually central, then there is a convergent representation ρ , conjugate of the latter, such that $H_0(M, \mathrm{Ad}_\rho)_v = 0$.*

Proof. Using twice the Universal Coefficient Theorem we obtain :

$$H_0(M, \text{Ad}_\rho)_v \otimes_{\mathcal{O}_v} \mathcal{O}_v/(t) \simeq H_0(M, \text{Ad}_{\bar{\rho}}) \simeq H^0(M, \text{Ad}_{\bar{\rho}})$$

By the preceding lemma, we can chose ρ such that $\bar{\rho}$ is not abelian, hence $H^0(M, \text{Ad}_{\bar{\rho}})$ is trivial and the lemma is proved. \square

Remark 1.33. In fact, if M is the complement of a knot in \mathbb{S}^3 , the condition that ρ is not residually central is empty : a reducible character $v \in Y$ corresponds to a reducible representation that is limit of irreducible representations. It is well-know since De Rham ([deR67, HPSP01]) that in this case the eigenvalue of $\bar{\rho}$, $\lambda : \pi_1(M) \rightarrow \mathbb{C}^*$, sends the square of any meridian m on a root of the Alexander polynomial. As the Alexander polynomial can not admit 1 as a root, this situation never occurs.

Lemma 1.34. *If v corresponds to an irreducible character, then $H_1(M, \text{Ad}_\rho) \simeq \Omega_{\bar{B}[\Gamma]/k} \otimes \mathcal{O}_v$.*

Proof. As in the proof of Proposition 1.22, there is a morphism of \mathcal{O}_v -modules $\Omega_{\bar{B}[\Gamma]/k} \otimes \mathcal{O}_v \rightarrow H_1(M, \text{Ad}_\rho)_v$. As ρ is residually irreducible, there exists $\alpha, \beta \in \Gamma$ such that $\text{Tr } \rho(\Delta_{\alpha, \beta}) \in \mathcal{O}_v^*$ hence the theorem of Saito again holds, and permits us to produce an inverse morphism as in [Mar15]. \square

2 The Reidemeister torsion form

General theory of the Reidemeister torsion is discussed in [Mil66], [GKZ94, Appendix A], [Por97, Chapter 0].

2.1 Definitions, and characterisation as a differential form

2.1.1 First definition

Given a finite complex C^* of k -vector spaces

$$C^0 \xrightarrow{d_0} C^1 \xrightarrow{d_1} \dots \xrightarrow{d_{n-1}} C^n$$

fix $\{c^i\}_{i=0\dots n}$ and $\{h^i\}_{i=0\dots n}$ families of basis of the C^i 's and the H^i 's, one can define the *torsion* of the based complex $\text{tor}(C^*, \{c^i\}, \{h^i\})$ to be the alternate product of the determinants of the base change induced by this choices. More precisely, we have the exact sequences

$$\begin{aligned} 0 \rightarrow Z^i \rightarrow C^i \rightarrow B^{i+1} \rightarrow 0 \\ 0 \rightarrow B^i \rightarrow Z^i \rightarrow H^i \rightarrow 0 \end{aligned}$$

that define B^i, Z^i and H^i . Pick a system of basis $\{b^i\}$ of the B^i 's, first one obtains a basis of Z^i for any i , given by a section $H^i \rightarrow Z^i$, and then a section $B^{i+1} \rightarrow C^i$ provides a basis of C^i : $b^i \sqcup \bar{h}^i \sqcup \bar{b}^{i+1}$, where the bars denote the image by the chosen sections. Now compare this new basis with c^i , and take the determinant of the matrix which exchange those basis, denoted by $[b^i \sqcup \bar{h}^i \sqcup \bar{b}^{i+1} : c^i]$. One can show that the alternate product of those determinants does not depend of the lifts and of the system $\{b^i\}$ and we define

$$\text{tor}(C^*, \{c^i\}, \{h^i\}) = \prod_i [b^i \sqcup \bar{h}^i \sqcup \bar{b}^{i+1} : c^i]^{(-1)^i} \in k^*/\{\pm 1\}$$

2.1.2 Second definition : torsion of an exact complex (Cayley formula)

If the complex is exact, one has the following alternative description : pick a system of basis $\{c^i\}$ of the C^i 's that induces, for each i , a splitting $C^i = \ker d_i \oplus K^i$, where K^i is a supplementary of $\ker d_i$ in C^i . Then each d_i restricts to an isomorphism $d|_{K^i} : K^i \rightarrow \ker d_{i+1}$, and we define

$$\text{tor}(C^*, \{c^i\}) = \prod_i \det(d|_{K^i})^{(-1)^i}$$

Again, it's defined up to sign as soon as we haven't fixed an order for the basis.

2.1.3 Third definition : the Euler isomorphism

Recall that the determinant of a n -dimensional vector space V is $\det(V) = \Lambda^n V$. One define the *determinant* of a complex $\det(C^*) = \bigotimes_i \det(C^i)^{\otimes (-1)^i}$. The cohomology of this complex is naturally graded by the degree, and we have the theorem.

Theorem 2.1. *There is a natural isomorphism*

$$\text{Eu} : \det(C^*) \xrightarrow{\sim} \det(H^*(C^*))$$

Proof. Again, we write the two exact sequences

$$0 \rightarrow Z^i \rightarrow C^i \rightarrow B^{i+1} \rightarrow 0$$

$$0 \rightarrow B^i \rightarrow Z^i \rightarrow H^i \rightarrow 0$$

Then the proof reduces to the particular case of a short exact sequence :

Lemma 2.2. *For an exact sequence of vector spaces*

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

one has an isomorphism

$$\det(A) \otimes \det(C) \simeq \det(B)$$

given by, for any choice of basis $\{a_1, \dots, a_m\}$ of A , $\{c_1, \dots, c_n\}$ of C ,

$$(a_1 \wedge \dots \wedge a_m) \otimes (c_1 \wedge \dots \wedge c_n) \mapsto a_1 \wedge \dots \wedge a_m \wedge \bar{c}_1 \wedge \dots \wedge \bar{c}_n$$

□

Definition 2.3. Given a complex C^* , and a system of basis $\{c^i\}$ of the C^i 's, $c^i = \{c_1^i, \dots, c_{n_i}^i\}$, then $\bigwedge c^i = c_1^i \wedge \dots \wedge c_{n_i}^i$ is a basis of $\det(C^i)$, and then we denote by $c = \bigotimes_i (\bigwedge c^i)^{\otimes (-1)^i}$ the induced basis of $\det(C^*)$.

Then the torsion of the based complex is defined by

$$\text{tor}(C^*, \{c^i\}) = \text{Eu}(c) \in \det(H^*(C^*))$$

2.1.4 The torsion of the twisted cellular complex

In the case of the twisted complex of M , we pick a basis of $\mathfrak{sl}_2(k)$, for example $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The cellular decomposition of M provides a $\mathbb{Z}[\pi_1(M)]$ -basis of $C_i(\tilde{M})$ for any i , denoted by $\{\tilde{e}_i^1, \dots, \tilde{e}_i^{n_i}\}$. We will denote by $f_{1,E}^i \in C^i(M, \text{Ad}_\rho)$ the map that sends \tilde{e}_i^1 on E and extend it in an Ad_ρ -equivariant way, and similarly we obtain a basis $f^i = \{f_{k,E}^i, f_{k,F}^i, f_{k,H}^i, k = 1 \dots n_i\}$ which is a basis of $C^i(M, \text{Ad}_\rho)$. As in the previous section, we denote by $c = \bigotimes_i (\Lambda f^i)^{\otimes (-1)^i}$, and we define the *Reidemeister torsion of the twisted complex*

$$\text{tor}(M, \rho) = \text{Eu}(c) \in \det(H^*(M, \text{Ad}_\rho))$$

Remark 2.4. It does not depend on a choice of the lifts of the cells e_i^k in the universal cover of M ; neither it depends on the conjugacy class of ρ .

2.1.5 Torsion and character variety

Theorem 2.5. *Let Y be (the smooth projective model of) a component of irreducible type of $\bar{X}(M)$, $\rho : \Gamma \rightarrow \text{SL}_2(k(Y))$ the tautological representation. Then the Reidemeister torsion $\text{tor}(M, \text{Ad}_\rho)$ -or $\text{tor}(M)$ - is a rational differential form on Y .*

Proof. From the definition, the torsion $\text{tor}(M)$ is an element of $\det(H^*(M, \text{Ad}_\rho)) = \det(H^0(M, \text{Ad}_\rho)) \otimes \det(H^1(M, \text{Ad}_\rho))^* \otimes \det(H^2(M, \text{Ad}_\rho))$. From Proposition 1.23, we deduce that $\det(H^0(M, \text{Ad}_\rho)) \simeq \det(H^2(M, \text{Ad}_\rho)) \simeq k(Y)$, and that $\det(H^1(M, \text{Ad}_\rho))^* \simeq H^1(M, \text{Ad}_\rho)^* \simeq H_1(M, \text{Ad}_\rho)$. Then Proposition 1.22 permits us to conclude that the latter is isomorphic to $\Omega_{k(Y)/k}$. \square

2.2 Examples and computations

We give a relation between the torsion as we defined above and the previous work of J.Porti and then J.Dubois in [Por97, Dub06, DHY09]. As the second author computed explicit formulae in many cases, this will permit us to perform direct computations in several examples.

Proposition 2.6 (see also Proposition 4.2 of [Por97]). *Given a finite point $v \in Y$, fix μ a simple closed curve in ∂M , then the following holds :*

$$\text{tor}(M, \text{Ad}_{\bar{\rho}}) = \frac{1}{\tau_\mu(M, \bar{\rho})} \frac{2}{\sqrt{(\text{Tr } \bar{\rho}(\mu))^2 - 4}} dY_\mu$$

where $\tau_\mu(M, \text{Ad}_{\bar{\rho}})$ is the torsion in the sense of Porti-Dubois at the point v .

Remark 2.7. Here we use the convention of [Por97], that corresponds to ours. In [Dub06] for instance, the torsion defined is the inverse of the latter.

Remark 2.8. Notice that τ_μ makes sense only when v is not a critical point of Y_μ . If we pull-back this formula on the augmented variety by setting $Y_\mu = Z_\mu + Z_{\mu^{-1}}$ then the formula becomes rational : $\text{tor}(M) = \frac{1}{\tau_\mu} \frac{dZ_\mu}{\mu}$.

Proof. The torsion in the sense of Porti-Dubois is defined pointwise as a function on a smooth open subset U of the character variety, and it is proved in [Por97] that it is in fact analytic. For $v \in U$, pick any generator P of $H^0(\partial M, \text{Ad}_{\bar{\rho}})$. Then one obtains basis of $H^i(M, \text{Ad}_{\bar{\rho}})$, $i = 1, 2$, as follows : since we assumed that $\bar{\rho}$ is not a critical point of the trace function Y_μ , the restriction map $H^1(M, \text{Ad}_{\bar{\rho}}) \rightarrow H^1(\mu, \text{Ad}_{\bar{\rho}})$ is an isomorphism. The latter is identified to k via $H^1(\mu, \text{Ad}_{\bar{\rho}}) \rightarrow k$, $f \mapsto \text{Tr}(Pf(\mu))$, and the composition of those maps is simply the differential dY_μ . Then the generator of $H^1(M, \text{Ad}_{\bar{\rho}})$ is fixed to be $dY_\mu^{-1}(1)$. For the generator of $H^2(M, \text{Ad}_{\bar{\rho}})$, pull-back 1 via the isomorphisms $H^2(M, \text{Ad}_{\bar{\rho}}) \rightarrow H^2(\partial M, \text{Ad}_{\bar{\rho}}) \xrightarrow{\phi_P} k$, where $\phi_P(f) = f(P)$. As P appears both in H^1 and in H^2 , the torsion does not depend on a choice of P . Finally, we have chosen H for the generator of $H^2(M, \text{Ad}_\rho)$, and the isomorphism $H_1(M, \text{Ad}_\rho) \rightarrow \Omega_{k(Y)/k}$ provides a term $\rho_0(\mu)$, we need to normalize by $\sqrt{\frac{\text{Tr}(H^2)}{\text{Tr}(\bar{\rho}_0(\mu)^2)}}$, and the result follows. \square

Example 2.9 (The trefoil knot). Recall that the tautological representation of the component of irreducible type $X \subset X(M)$ is given by the formulas :

$$\rho(a) = \begin{pmatrix} t & 1 \\ -(t^2 + 1) & -t \end{pmatrix}, \rho(b) = \begin{pmatrix} -j & 0 \\ 0 & -j^2 \end{pmatrix}$$

In [Dub06], for any boundary curve μ , $\tau_\mu(\bar{\rho})$ is a constant k that does not depend on $\bar{\rho}$. Take μ the meridian ab^{-1} , $Y_\mu = (j - j^2)t$, $Z_\mu = u$, then

$$\text{tor}(M) = k \frac{du}{u}$$

It has no zeros, and two poles at zero and infinity, as expected its divisor's degree is -2.

Example 2.10 (The figure-eight knot). Here we take μ to be the longitude of M , denote its trace by $Y_\mu = x^4 - 5x^2 + 2$, then $\tau_\mu(x, y) = \frac{1}{5-2x^2}$ and one obtains $\text{tor}(M) = \frac{dZ_\mu}{(5-2x^2)Z_\mu}$. A careful examination shows that it has no poles, and zeros only at infinity : take $x = 1/t$ a local coordinate, $\frac{dZ_\mu}{Z_\mu} = \frac{dY_\mu}{\sqrt{Y_\mu^2 - 4}} \simeq \frac{dt}{t}$, hence each of the four ideal points contribute as a zero of order 1. The divisor's degree of the torsion is 4, that confirms the fact that Y has genus 3.

Example 2.11 (The knot 5.2). Here we develop this example from [DHY09]. The fundamental group is isomorphic to $\pi_1(M) = \langle u, v \mid vw = wu \rangle$ where $w = u^{-1}v^{-1}uvu^{-1}v^{-1}$. The component of irreducible type of the character variety is described by the Riley polynomial $\phi(S, U)$. In our setting, with $x = \text{Tr } u = \text{Tr } v$ and $y = \text{Tr } uv$, then $x = S^{\frac{1}{2}} + S^{-\frac{1}{2}}$ and $y = S + S^{-1} - U$, we obtain

$$X^{\text{irr}}(M) = \{(x, y) \in k^2 \mid -x^2(y-1)(y-2) + y^3 - y^2 - 2y + 1 = 0\}$$

This affine curve compactifies with two points at infinity : an ordinary double point $x = \infty, y = 1$ or $y = 2$, and a simple point $x = y = \infty$. Apart from this, the variety is smooth. By the Noether-Plücker formula, its genus is $(d-1)(d-2)/2 - \delta$, with $d = 4$ and $\delta = 1$, hence $g(X) = 2$.

The extension $\alpha + \alpha^{-1} = x$ gives a $2 : 1$ map $Y \rightarrow X$, that ramifies in $x^2 = 4$. The Hurewitz formula implies $\chi(Y) = 2\chi(X) - 6 = -10$, hence Y is a genus 6 curve.

From [DHY09] again, with μ a longitude, $\tau_\mu = 5x^4(y-2) - x^2(5y^2+7y-31) + 7(y^2-y-3)$, and $Y_\mu = (y^3-6y^2+12y-8)x^{10} - (3y^4-10y^3-y^2-68)x^8 + 3(y^5-43y^3+48y^2+86y-116)x^6 + (y^6+6y^5-23y^4-28y^3+96y^2+28y-105)x^4 + (2y^6-y^5-16y^4+6y^3+40y^2-9y-34)x^2 + 2$. As $\text{tor}(M) = \frac{dY_\mu}{\tau_\mu \sqrt{Y_\mu^2-4}}$, we compute the vanishing order of the torsion at the 3 different ideal points :

1. $x \sim \frac{1}{t}$, $y \sim 1+t^2$, then $\tau_\mu \sim \frac{1}{t^4}$, $\frac{dY_\mu}{\sqrt{Y_\mu^2-4}} \sim \frac{dt}{t}$ and $\text{tor} \sim t^3 dt$
2. $x \sim \frac{1}{t}$, $y \sim 2+3t$, then $\tau \sim \frac{1}{t^2}$, $\frac{dY_\mu}{\sqrt{Y_\mu^2-4}} \sim \frac{dt}{t}$ and $\text{tor} \sim t dt$
3. $x \sim \frac{1}{t(1-2t^2)}$, $y \sim \frac{1}{t^2(1-2t^2)}$, then again $\tau \sim \frac{1}{t^2}$, $\frac{dY_\mu}{\sqrt{Y_\mu^2-4}} \sim \frac{dt}{t}$ and $\text{tor} \sim t dt$

Finally, notice that $Y \rightarrow X$ does not ramify at infinity, hence to each ideal point of X correspond 2 ideal points of Y , and the divisor's degree of tor on Y is 10, as expected.

Example 2.12 (The knot 6.1). This example arises from [DHY09] too. $\pi_1(M) = \langle u, v \mid vw = wu \rangle$ where $w = (vu^{-1}v^{-1}u)^2$. The irreducible type part of the character variety is

$$X = \{(x, y) \in \mathbb{C}^2 \mid x^4(y-2)^2 - x^2(y+1)(y-2)(2y-3) + (y^3-3y-1)(y-1) = 0\}$$

The two ideal points are non ordinary double points :

1. When $y \rightarrow 2$, $x \rightarrow \infty$, we have a double point of type " $y^2 - x^6$ ", its δ -invariant is 3.
2. When $w, x \rightarrow \infty$, we have a double point of type " $y^2 - x^8$ ", its δ -invariant is 4.

Hence $g(X) = (d-1)(d-2)/2 - \sum \delta_i = 10 - 3 - 4 = 3$. The covering map $Y \rightarrow X$ given by $\alpha + \alpha^{-1} = x$ ramifies in eight finite points, thus $\chi(Y) = -16$

When desingularizing X one obtains four ideal points, the same kind of computations as in Example 2.11 are shortened as follows :

1. $x \sim \frac{1}{t(1+at^2)}$, $y \sim \frac{2}{1+at^2}$ with a a root of $4a^2 + 6a + 1$ then in both cases $\tau \sim \frac{1}{t^2}$, $\frac{d\lambda}{\lambda} \sim \frac{dt}{t}$ and $\text{tor} \sim t dt$
2. $x \sim \frac{1}{t(1-t^2)}$, $y \sim \frac{1}{t^2(1-t^2)}$, then $\tau \sim \frac{1}{t^6}$, $\frac{d\lambda}{\lambda} \sim \frac{dt}{t}$ and $\text{tor} \sim t^5 dt$
3. $x \sim \frac{1}{t(1-2t^2+6t^4-25t^6)}$, $y \sim \frac{1}{t^2(1-2t^2+6t^4-25t^6)}$, then $\tau \sim 1$, $\frac{d\lambda}{\lambda} \sim t dt$ and $\text{tor} \sim t dt$

Then notice that $Y \rightarrow X$ is not ramified at infinity, thus the divisor's degree of tor is 16, as expected.

3 The torsion at a finite point

Throughout this section $v \in Y$ is a finite point, and $\rho : \Gamma \rightarrow \text{SL}_2(\mathcal{O}_v)$ is the tautological representation. In the sequel we will assume that ρ is not residually central, see Remark 1.33. We prove the following :

Theorem 3.1 (Main Theorem 1). *The differential form $\text{tor}(M)$ vanishes at v with order the lenght of the torsion part in the \mathcal{O}_v -module $H_1(M, \text{Ad}_\rho)_v$. In particular tor has no poles nor zeros if v projects on a smooth point of $X(M)$.*

3.1 Proof of Main Theorem 1

Definition 3.2. We will say a complex of \mathcal{O}_v -modules C^* to be *rationally exact* if $C^* \otimes k(Y)$ is an exact complex.

The following theorem is the key argument in the proof of Main Theorem 1, see [GKZ94, Appendix A, theorem 30] :

Theorem 3.3. *Let v be a valuation on $k(Y)$, and C^* a rationally exact based complex of \mathcal{O}_v -modules, with basis $\{c^i\}_i$. Then the following holds :*

$$v(\text{tor}(C^* \otimes k(Y), \{c^i\}_i)) = \sum_k (-1)^k \text{lenght}(H^k(C^*))$$

We will denote by t an uniformizing parameter of \mathcal{O}_v , that is an element $t \in k(Y)$ with $v(t) = 1$. Hence the one-dimensional vector-space $\Omega_{k(Y)/k}$ is generated by dt . In the sequel we denote by $f \in k(Y)$ the function such that $\text{tor}(M) = fdt$ (more precisely, $\text{tor}(M) = fdt \otimes H^*$). The strategy is to construct a rationally exact based complex of \mathcal{O}_v modules C^* with $\text{tor}(C^*) = f$, and then use Theorem 3.3.

Let us denote by D^* the trivial complex

$$0 \rightarrow \Omega_{\mathcal{O}_v/k}^* \xrightarrow{0} H^0(M, \text{Ad}_\rho)_v$$

Lemma 3.4. *The \mathcal{O}_v -module $\Omega_{\mathcal{O}_v/k}$ is free of rank one, with dt as a generator.*

Proof. Its rank is one because $\Omega_{\mathcal{O}_v/k} \otimes k(Y)$ is one-dimensional. it is free because $\Omega_{\mathcal{O}_v/k} \otimes k \simeq k$, as it is the co-tangent space at a smooth point $v \in Y$. The fact that dt generates is a local fact, hence it is sufficient to consider $k(Y) \simeq k(t)$, $\mathcal{O}_v \simeq k[t]_{(0)}$ and the result is clear. \square

Let $\alpha : C^1(M, \text{Ad}_\rho)_v \rightarrow \Omega_{\mathcal{O}_v}^*$ defined by $\alpha(f)(d(Y_\gamma \otimes 1)) = \text{Tr}(f(\gamma)\rho_0(\gamma))$, and $\beta : C^2(M, \text{Ad}_\rho)_v \rightarrow H^2(M, \text{Ad}_\rho)_v \rightarrow H^2(\partial M, \text{Ad}_\rho)_v \rightarrow H^0(\partial M, \text{Ad}_\rho)_v$ is the composition of the reduction mod $\text{im } d$, the restriction map and the Poincare duality.

Proposition 3.5. *The maps α and β induce a morphism of complexes of \mathcal{O}_v -modules $\phi : C^*(M, \text{Ad}_\rho)_v \rightarrow D^*$ that is rationally a quasi-isomorphism.*

Proof. Let us draw the diagram

$$\begin{array}{ccccc} 0 & \xrightarrow{0} & \Omega_{\mathcal{O}_v/k}^* & \xrightarrow{0} & H^0(\partial M, \text{Ad}_\rho)_v^* \\ \uparrow 0 & & \uparrow \alpha & & \uparrow \beta \\ C^0(M, \text{Ad}_\rho)_v & \xrightarrow{d} & C^1(M, \text{Ad}_\rho)_v & \xrightarrow{d} & C^2(M, \text{Ad}_\rho)_v \end{array}$$

First we need to show that it commutes. It is clear from the definition that $\beta \circ d = 0$, and for any $\zeta \in C^0(M, \text{Ad}_\rho)_v$, for any $\gamma \in \Gamma$, $\alpha(d\zeta)d(Y_\gamma \otimes Z_\lambda) = \text{Tr}(d\zeta(\gamma)\rho_0(\gamma))$. As $d\zeta(\gamma) = \rho(\gamma)\zeta\rho^{-1}(\gamma) - \zeta$ and as for any $\xi \in \text{sl}_2(\mathcal{O}_v)$, $\text{Tr}(\xi\rho_0(\gamma)) = \text{Tr}(\xi\rho(\gamma))$, the latter is $\text{Tr}[\rho_0(\gamma), \zeta]$, hence it is zero.

Now we prove that the complexes $C^*(M, \text{Ad}_\rho)$ and $D^* \otimes k(Y)$ are quasi-isomorphic. We have $H^1(M, \text{Ad}_\rho) \simeq H_1(M, \text{Ad}_\rho)^* \simeq \Omega_{k(Y)/k}^* \simeq \Omega_{\mathcal{O}_v/k}^* \otimes k(Y)$, the first isomorphism comes from the Universal Coefficient Theorem, the second from Theorem 1.22, and the third is a classical fact of algebraic geometry, see [Liu06, Chapter 6]. The very same proof as the end of the argument in Proof of Proposition 1.23 permits us to conclude that $H^2(M, \text{Ad}_\rho)_v \simeq H^0(\partial M, \text{Ad}_\rho)_v^*$. As ρ is not abelian, $H^0(M, \text{Ad}_\rho)_v = 0$ and this concludes the proof. \square

Definition 3.6. The *cone* of the morphism of complexes ϕ is defined as the complex $D^* \oplus C^{*+1}(M, \text{Ad}_\rho)_v$:

$$C^0(M, \text{Ad}_\rho)_v \xrightarrow{d} C^1(M, \text{Ad}_\rho)_v \xrightarrow{d, \alpha} C^2(M, \text{Ad}_\rho)_v \oplus \Omega_{\mathcal{O}_v/k}^* \xrightarrow{\beta} H^0(\partial M, \text{Ad}_\rho)_v^*$$

The preceding lemma asserts that the complex $\text{Cone}(\phi)$ is rationally exact. Moreover, it is naturally a based complex, by the natural basis of $C^*(M, \text{Ad}_\rho)_v$, by the duals of $dt \in \Omega_{\mathcal{O}_v/k}$ and of $H \in H^0(\partial M, \text{Ad}_\rho)_v^*$. Then the torsion of this complex is

$$\text{tor}(\text{Cone}(\phi) \otimes k(Y)) = \frac{\text{tor}(D^* \otimes k(Y))}{\text{tor}(C^*(M, \text{Ad}_\rho))} \quad (1)$$

We deduce the following lemma :

Lemma 3.7. *The torsion of the complex $\text{Cone}(\phi) \otimes k(Y)$ is $\frac{1}{f} \in k(Y)$*

Proof. By construction $\text{tor}(D^* \otimes k(Y)) = dt \otimes H^*$, and the result follows from formula (1). \square

Now we can apply Theorem 3.3 to the rationally exact complex $\text{Cone}(\phi)$, and obtain $v(f) = \sum_i (-1)^{i+1} \text{lenght}(H^i(\text{Cone}(\phi)))$. Now we compute the cohomology of this complex. The exact sequence $0 \rightarrow D^* \rightarrow \text{Cone}(\phi) \rightarrow C^{*+1}(M, \text{Ad}_\rho)_v \rightarrow 0$ induces the long exact sequence in cohomology : $0 \rightarrow H^0(\text{Cone}(\phi)) \rightarrow H^1(M, \text{Ad}_\rho)_v \xrightarrow{\alpha} \Omega_{\mathcal{O}_v/k}^* \rightarrow H^1(\text{Cone}(\phi)) \rightarrow H^2(M, \text{Ad}_\rho)_v \rightarrow H^0(\partial M, \text{Ad}_\rho)_v^* \rightarrow H^2(M, \text{Ad}_\rho)_v \rightarrow H^2(\text{Cone}(\phi)) \rightarrow 0$

Lemma 3.8. *The morphism α is surjective.*

Proof. We construct a section $s : \Omega_{\mathcal{O}_v/k}^* \rightarrow H^1(M, \text{Ad}_\rho)_v$ as follows : let $\theta : \Omega_{\mathcal{O}_v/k} \rightarrow \mathcal{O}_v$ a morphism of \mathcal{O}_v -modules, equivalently, by the universal property of Ω , θ is a k -derivation $\mathcal{O}_v \rightarrow \mathcal{O}_v$. We define $s(\theta) \in H^1(M, \text{Ad}_\rho)_v$ by the formula $s(\theta)(\gamma) = \theta(\rho(\gamma))\rho(\gamma)^{-1}$ where by $\theta(\rho(\gamma))$ we mean that we apply θ to each coefficient of $\rho(\gamma)$. We compute directly that $\text{Tr } s(\theta(\gamma)) = \theta(\det \rho(\gamma)) = 0$, and that $s(\theta(\gamma\delta)) = s(\theta(\gamma)) + \text{Ad}_{\rho(g)} s(\theta(\delta))$. Then $\alpha \circ s(\theta)(Y_\gamma) = \text{Tr}(\theta(\rho(\gamma))) = \theta(Y_\gamma)$, and the lemma is proved. \square

Lemma 3.9. *The morphism α is injective, hence $H^0(\text{Cone}(\phi)) = 0$.*

Proof. By the Universal Coefficient Theorem, $H^1(M, \text{Ad}_\rho)_v \simeq H_1(M, \text{Ad}_\rho)_v^* \simeq \mathcal{O}_v$ because $H_1(M, \text{Ad}_\rho) \simeq k(Y)$. The lemma follows. \square

Lemma 3.10. *Denote by T the torsion part of the module $H_1(M, \text{Ad}_\rho)_v$. The \mathcal{O}_v -module $H^1(\text{Cone}(\phi))$ is isomorphic to the torsion module T .*

Proof. Again by Universal Coefficient Theorem, there is an isomorphism

$$H^2(M, \text{Ad}_\rho)_v \simeq H_2(M, \text{Ad}_\rho)_v^* \oplus \text{Ext}(H_1(M, \text{Ad}_\rho)_v, \mathcal{O}_v) \simeq \mathcal{O}_v \oplus T$$

As α is surjective and $H^0(\partial M, \text{Ad}_\rho)_v^* \simeq \mathcal{O}_v$, we deduce that $H^1(\text{Cone}(\phi)) = \ker(H^2(M, \text{Ad}_\rho)_v \rightarrow H^0(\partial M, \text{Ad}_\rho)_v^*) \simeq T$ \square

Lemma 3.11. *The map $H^2(M, \text{Ad}_\rho)_v \rightarrow H^0(\partial M, \text{Ad}_\rho)_v^*$ is surjective, hence $H^2(\text{Cone}(\phi)) = 0$.*

Proof. Under the isomorphism $H^2(\partial M, \text{Ad}_\rho)_v \simeq H^0(\partial M, \text{Ad}_\rho)_v^*$, this map is just the last one of the long exact sequence of the pair $(M, \partial M)$. \square

Proof of Main Theorem 1. Now we just have to fit together the arguments : write $\text{tor}(M) = f dt$ near v , the vanishing order of $\text{tor}(M)$ at v is given by $v(f) = -\text{tor}(\text{Cone}(\phi) \otimes k(Y)) = \text{lenght}(H^1(\text{Cone}(\phi))) = \text{lenght}(T)$. \square

3.2 Interpretation of the Theorem : singularities and Alexander module

The aim of this section is to provide a geometric signification to the length of the module $H_1(M, \text{Ad}_\rho)_v$ that appears in the statement of the Main Theorem 1.

Notice that if v corresponds to an irreducible character, $H_1(M, \text{Ad}_\rho)_v \simeq \Omega_{\bar{B}[\Gamma]/k} \otimes \mathcal{O}_v$. If $\nu(v)$ is a smooth point, the latter is isomorphic to the localized of the module of differentials at v : $H_1(M, \text{Ad}_\rho)_v \simeq \Omega_{\mathcal{O}_v/k}$, and this is simply \mathcal{O}_v . Hence we are interested by the cases when $\nu(v)$ is not a smooth point of Y , or does correspond to a reducible character. This case is a singular case too : here v corresponds in $X(M)$ to an intersection point of a component of irreducible type and the reducible component. We treat both cases.

3.2.1 Singularities at irreducible characters

One has the following exact sequence $0 \rightarrow T \rightarrow \Omega_{\bar{B}[\Gamma]/k} \otimes \mathcal{O}_v \rightarrow \Omega_{\mathcal{O}_v/k}$ provided by the normalization map ν . Thus $\text{lenght}(T)$ is an invariant of the branch of v at the singularity $\nu(v)$. We do not know any general formula, but we are able to compute it directly if a curve equation is given. Notice that this general question relies on the (still open) following problem : given a point of a curve $x \in X$, is it true that x is smooth iff $\Omega_{\mathcal{O}_x/k} \simeq \mathcal{O}_x$? See [Ber94] for a survey of the topic. We treat the example of plane singularities.

Assume $x = (0, 0) \in k^2$, and C is the curve defined by the polynomial $X^p - Y^q$, with $p < q$. The singular point x has multiplicity p ; pick \tilde{x} a pre-image of x by the normalization $\nu : \tilde{C} \rightarrow C$, and denote its discrete valuation ring by \mathcal{O} . Denote by $n = \gcd(p, q)$, $p' = \frac{p}{n}$, $q' = \frac{q}{n}$. The normalization $\nu : \mathbb{A}_k^1 \rightarrow C$ is given by

$$k[X, Y]/(X^p - Y^q) \rightarrow k[S]$$

$$X \mapsto S^{q'}, Y \mapsto S^{p'}$$

We compute $\Omega_{\mathcal{O}_x/k} = \mathcal{O}_x dX \oplus \mathcal{O}_x dY / (pX^{p-1}dX - qY^{q-1}dY)$, thus $\Omega_{\mathcal{O}_x/k} \otimes \mathcal{O} = \mathcal{O}dX \oplus \mathcal{O}dY / (pS^{q'(p-1)}dX - qS^{p'(q-1)}dY)$. The morphism $\Omega_{\mathcal{O}_x/k} \otimes \mathcal{O} \rightarrow \Omega_{\mathcal{O}/k}$ sends dX onto $q'S^{q'-1}dS$ and dY onto $p'S^{p'-1}dS$. The kernel of this morphism is generated by $p'dX - q'S^{q'-p'}dY \in \Omega_{\mathcal{O}_x/k} \otimes \mathcal{O}$, and its annihilator is $(nS^{q'(p-1)})$. Hence $T \simeq \mathcal{O}/(S^{q'(p-1)})$ and $\text{lenght}(T) = q'(p-1)$.

3.2.2 Singularities at reducible character

In this section we focus on $v \in Y$ that corresponds to a reducible character. It is the case precisely when it corresponds to an intersection point of the component of irreducible type corresponding to Y with the reducible component.

We need to study the \mathcal{O}_v -module $H_1(M, \text{Ad}_\rho)_v$, but here we do not dispose of an interpretation in term of the cotangent space. The strategy is the following : recall that we can write $H_1(M, \text{Ad}_\rho)_v = \mathcal{O}_v \oplus \bigoplus_i \mathcal{O}_v/(t^{n_i})$, so we need to compute $\sum_i n_i$. First we consider $H_1(M, \text{Ad}_\rho)_v \otimes \mathcal{O}_v/(t) \simeq H_1(M, \text{Ad}_{\bar{\rho}})$, it is a k -vector space whose dimension will give the number of i 's in the latter sum. Then we prove that under some hypothesis, all of the n_i 's are equal to 1.

Up to conjugacy, we know that we can fix the tautological representation $\rho : \Gamma \rightarrow \text{SL}_2(\mathcal{O}_v)$ such that $\forall \gamma \in \Gamma, \bar{\rho}(\gamma) = \begin{pmatrix} \lambda(\gamma) & \lambda^{-1}(\gamma)u(\gamma) \\ 0 & \lambda^{-1}(\gamma) \end{pmatrix}$.

Let $\varphi : \Gamma \rightarrow \mathbb{Z}$ a choice of an abelianisation, and m a meridian, i.e. an element of Γ such that $\varphi(m) = 1$. Then for all $\gamma \in \Gamma$, $\lambda(\gamma) = \lambda(m)^{\varphi(\gamma)}$. When the context will be clear, we will denote by $\lambda = \lambda(m)$, notice that it does not depend on the choice of m , up to exchange λ with λ^{-1} .

The following is due to de Rham :

Theorem 3.12. [deR67, HPSP01] *If ρ is residually reducible, then λ^2 is a root of the Alexander polynomial $\Delta(M)$.*

As it is well-known that $\Delta(1) \neq 0$, we proved the following :

Proposition 3.13. *If $H^1(M) = \mathbb{Z}$, then for any v that corresponds to a reducible character, the tautological representation is not residually central. Consequently it is conjugated to a convergent representation that is not residually abelian.*

Now we assume that ρ is not residually abelian. We will denote by $C^*(M, k_{\lambda^{\pm 2}})$ the complex of group cohomology of Γ with coefficients in k twisted by the action of λ^2 , resp. λ^{-2} .

Lemma 3.14. *The map $u : \Gamma \rightarrow k$ is a non trivial cocycle in $H^1(M, k_{\lambda^2})$.*

Proof. First check that, as $\bar{\rho}$ is a morphism, one has $u(\gamma\delta) = u(\gamma) + \lambda^2(\gamma)u(\delta)$ for all $\gamma, \delta \in \Gamma$, and thus $u \in H^1(M, k_{\lambda^2})$.

Now assume $u(\gamma) = \lambda^2(\gamma) - 1$ is a coboundary. Then

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda(\gamma) & \lambda^{-1}(\gamma)u(\gamma) \\ 0 & \lambda^{-1}(\gamma) \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \lambda(\gamma) & 0 \\ 0 & \lambda^{-1}(\gamma) \end{pmatrix}$$

and $\bar{\rho}$ is abelian, a contradiction. □

Notice that the adjoint action of $\bar{\rho}$ on $\text{sl}_2(k)$ has the following matrix in the basis $\{E, H, F\}$:

$$\text{Ad}_{\bar{\rho}} = \begin{pmatrix} \lambda^2 & -2u & -\lambda^{-2}u^2 \\ 0 & 1 & \lambda^{-2}u \\ 0 & 0 & \lambda^{-2} \end{pmatrix} \quad (2)$$

The Γ -module $\text{sl}_2(k)$ will be denoted by $\text{Ad}_{\bar{\rho}}$, and K will be the submodule $E.k \oplus H.k$. In the sequel we will consider (co)homology groups of the group Γ with coefficients in

different Γ -modules. From now on we will not specify the " Γ " (or the " M ") in the label of the groups. From the matrix (2) we obtain the splitting :

$$0 \rightarrow K \rightarrow \text{Ad}_{\bar{\rho}} \rightarrow k_{\lambda^2} \rightarrow 0$$

that gives rise to the long exact sequence :

$$0 \rightarrow H^1(K) \rightarrow H^1(\text{Ad}_{\bar{\rho}}) \rightarrow H^1(k_{\lambda^2}) \rightarrow H^2(K) \rightarrow H^2(\text{Ad}_{\bar{\rho}}) \rightarrow H^2(k_{\lambda^2}) \rightarrow 0$$

Notice that $H^0(k_{\lambda^2})$ is trivial as soon as there exists $\gamma \in \Gamma$ such that $\lambda^2(\gamma) \neq 1$, i.e. as soon as $\bar{\rho}$ is not central as a character.

Similarly, the Γ module K splits as :

$$0 \rightarrow k_{\lambda^2} \rightarrow K \rightarrow k \rightarrow 0$$

hence the sequence

$$0 \rightarrow H^0(k) \rightarrow H^1(k_{\lambda^2}) \rightarrow H^1(K) \rightarrow H^1(k) \xrightarrow{\delta} H^2(k_{\lambda^2}) \rightarrow H^2(K)$$

First we study the map δ :

Lemma 3.15. *The map $\delta : H^1(k) \rightarrow H^2(k_{\lambda^2})$ is given by $\delta(\varphi) = 2u \cup \varphi$, where φ is the generator of $H^1(k)$ given by the abelianisation.*

Proof. We detail here a very classical argument of computation of the so-called "connection operator" in a long exact sequence of cohomology. Draw the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & C^1(k_{\lambda^2}) & \longrightarrow & C^1(K) & \longrightarrow & C^1(k) \longrightarrow 0 \\ & & \downarrow d & & \downarrow d & & \downarrow d \\ 0 & \longrightarrow & C^2(k_{\lambda^2}) & \longrightarrow & C^2(K) & \longrightarrow & C^2(k) \longrightarrow 0 \end{array}$$

with exact rows.

Pick $\varphi \in C^1(k)$ such that $d\varphi = 0$. Take $\tilde{\varphi} \in C^1(K)$ defined by $\tilde{\varphi}(\gamma) = \begin{pmatrix} \varphi(\gamma) & 0 \\ 0 & -\varphi(\gamma) \end{pmatrix}$. Then compute $d\tilde{\varphi}(\gamma, \delta) = \tilde{\varphi}(\gamma\delta) - \tilde{\varphi}(\gamma) - \gamma \cdot \tilde{\varphi}(\delta)$. Observe that by the formula (2) we have $\gamma \cdot \tilde{\varphi}(\delta) = \text{Ad}_{\bar{\rho}(\gamma)|_K} \tilde{\varphi}(\delta) = \tilde{\varphi}(\delta) + \begin{pmatrix} 0 & -2u(\gamma)\varphi(\delta) \\ 0 & 0 \end{pmatrix}$. As $\varphi \in Z^1(k)$, we get that $\tilde{\varphi}(\gamma\delta) = \tilde{\varphi}(\gamma) + \tilde{\varphi}(\delta)$, thus

$$d\tilde{\varphi}(\gamma, \delta) = \begin{pmatrix} 0 & -2u(\gamma)\varphi(\delta) \\ 0 & 0 \end{pmatrix}$$

The conclusion follows. \square

Definition 3.16. The abelianization morphism $\varphi : \Gamma \rightarrow \mathbb{Z}$ induces a structure of Γ -module on $k[t^{\pm 1}]$, given by $\gamma \cdot P = t^{\varphi(\gamma)} P$. The *Alexander module* is the first homology group of M with φ -twisted coefficients $k[t^{\pm 1}]$, we denote it by $H_1(M, k[t^{\pm 1}]_{\varphi})$, or shortly $H_1(k[t^{\pm 1}]_{\varphi})$. It is a torsion $k[t^{\pm 1}]$ -module.

The multiplication by $t - \lambda^2$ induces the splitting of Γ -modules :

$$0 \rightarrow k[t^{\pm 1}]_{\varphi} \xrightarrow{t-\lambda^2} k[t^{\pm 1}]_{\varphi} \rightarrow k_{\lambda^2} \rightarrow 0 \quad (3)$$

Lemma 3.17. *The sequence above induces the following long exact sequence :*

$$0 \rightarrow H^1(k_{\lambda^2}) \rightarrow H_1(k[t^{\pm 1}]_{\varphi}) \rightarrow H_1(k[t^{\pm 1}]_{\varphi}) \rightarrow H^2(k_{\lambda^2}) \rightarrow 0 \quad (4)$$

Proof. Let's write the long exact sequence in homology induced by (3)

$$\dots \rightarrow H^1(k_{\lambda^2}) \rightarrow H^2(k[t^{\pm 1}]_{\varphi}) \rightarrow H^2(k[t^{\pm 1}]_{\varphi}) \rightarrow H^2(k_{\lambda^2}) \rightarrow \dots$$

First, let's prove that the left and the right hand side of this sequence are 0's. We remark that the missing term on the right is $H^3(k[t^{\pm 1}]) \simeq 0$. Then as the term on the left is $H^1(k[t^{\pm 1}]_{\varphi}) \simeq \text{Ext}(H_0(k[t^{\pm 1}]_{\varphi}), k[t^{\pm 1}]) \simeq k[t^{\pm 1}]/(t-1)$ by the universal coefficients theorem, we obtain that the map induced by multiplication by $t - \lambda^2$

$$H^1(k[t^{\pm 1}]_{\varphi}) \xrightarrow{\cdot(t-\lambda^2)} H^1(k[t^{\pm 1}]_{\varphi})$$

is surjective, thus the first map on the left is $H^1(k[t^{\pm 1}]_{\varphi}) \xrightarrow{0} H^1(k_{\lambda^2})$.

Now, we have the universal coefficients theorem that gives an isomorphism $H^2(k[t^{\pm 1}]_{\varphi}) \simeq \text{Ext}(H_1(k[t^{\pm 1}]_{\varphi}), k[t^{\pm 1}])$. But as $H_1(k[t^{\pm 1}]_{\varphi})$ is a torsion module, the latter is $H_1(k[t^{\pm 1}]_{\varphi})$ itself, that ends the proof of the lemma. \square

We denote by $\theta : H^1(k_{\lambda^2}) \rightarrow H^2(k_{\lambda^2})$ the composition of the first and the third map in the sequence (4).

Lemma 3.18. $\theta(z) = \lambda^{-2}\varphi_{\cup}z$, in particular $2\lambda^2\theta(u) = \delta(\varphi)$.

Proof. Again, consider the sequence :

$$0 \rightarrow H^1(k_{\lambda^2}) \rightarrow H^2(k[t^{\pm 1}]_{\varphi}) \rightarrow H^2(k[t^{\pm 1}]_{\varphi}) \rightarrow H^2(k_{\lambda^2}) \rightarrow 0$$

Then the same kind of computation as in the proof of Lemma 3.15 gives the result. We sketch the proof : given $z \in H^1(k_{\lambda^2})$, first lift it as $\tilde{z}(t) \in C^1(k[t^{\pm 1}]_{\varphi})$, then send it on $\frac{\delta \tilde{z}(t)}{t-\lambda^2} \in H^2(k[t^{\pm 1}]_{\varphi})$, and then take the evaluation at $t = \lambda^2$. The result is precisely $\frac{d}{dt} \Big|_{t=\lambda^2} \delta \tilde{z}(t) = \lambda^{-2}\varphi_{\cup}z \in H^2(k_{\lambda^2})$ \square

We consider the operator $A : H_1(k[t^{\pm 1}]_{\varphi}) \rightarrow H_1(k[t^{\pm 1}]_{\varphi})$ induced by multiplication by $t - \lambda^2$. Assume that λ^2 is a root of order r of the Alexander polynomial, and let's write the Alexander module $H_1(M, k[t^{\pm 1}]_{\varphi})$ as a direct sum $\bigoplus_{i=1}^n k[t^{\pm 1}]/(P_i)$, with $P_i | P_{i+1}$. The Alexander polynomial is defined as $\prod_i P_i$, so there is i such that $\forall j \geq i, (t - \lambda^2) | P_j$. Then we claim the following fact : θ is an isomorphism iff the kernel of A , which is $H^1(k_{\lambda^2})$, has dimension r or in other words, iff $(t - \lambda^2)^2$ does not divide any of the P_i 's.

Definition 3.19. Define the k -th Alexander polynomial to be the greatest common divisor of the products of $n+1-k$ of the P_i 's. The Alexander polynomial is then the first Alexander polynomial in this sense.

We have proved the following lemma :

Lemma 3.20. *The morphism $\theta : H^1(k_{\lambda^2}) \rightarrow H^2(k_{\lambda^2})$ is an isomorphism iff λ^2 is a root of the r -th Alexander polynomial.*

Definition 3.21. We define the *cup-bracket*:

$$\begin{aligned} [\cup] : H^1(\Gamma, \text{Ad}_{\bar{\rho}}) \times H^1(\Gamma, \text{Ad}_{\bar{\rho}}) &\rightarrow H^2(\Gamma, \text{Ad}_{\bar{\rho}}) \\ (u, v) &\mapsto ([u \cup v] : (\gamma, \delta) \mapsto [u(\gamma), \text{Ad}_{\bar{\rho}(\gamma)} v(\delta)]) \end{aligned}$$

We define the first order deformation of $\bar{\rho}$ as the cocycle $v \in H^1(M, \text{Ad}_{\bar{\rho}})$ such that in $\text{SL}_2(\mathcal{O}_v/(t^2))$, the mod (t^2) -reduced tautological representation is given by $\rho' = (\text{Id} + tv)\bar{\rho}$. It is a well known fact of deformation theory that $[v, v] = 0$ in this context, in other words, v is in the kernel of the operator $B : H^1(\text{Ad}_{\bar{\rho}}) \rightarrow H^2(\text{Ad}_{\bar{\rho}})$ that sends a cocycle u on the cup-bracket $[u, v]$. The theorem is the following :

Theorem 3.22. *Let $v \in Y$ a finite point, assume that the tautological representation $\rho : \Gamma \rightarrow \text{SL}_2(\mathcal{O}_v)$ is residually reducible but not central, and that λ^2 is a root of order $r \geq 1$ of the Alexander polynomial $\Delta(M)$, and a root of the r -th Alexander polynomial. Assume moreover that the operator B has maximal rank, i.e. $\ker B = k.v$. Then the torsion form vanishes at order $2r - 2$ at v .*

Remark 3.23. In the generic case when λ^2 is a simple root, then the technical assumptions are automatically satisfied and the theorem always holds : thus in this case $\text{tor}(M)$ does not vanish at v .

Proof. We know that $\dim H^1(k_{\lambda^2}) = r$, and the map $\theta : H^1(k_{\lambda^2}) \rightarrow H^2(k_{\lambda^2})$ defined above is an isomorphism ; hence δ is injective and $\dim H^1(K) = r - 1$, $\dim H^1(\text{Ad}_{\bar{\rho}}) = 2r - 1$. Consider the sequence of Γ -modules

$$0 \rightarrow \text{sl}_2(\mathcal{O}_v/(t)) \rightarrow \text{sl}_2(\mathcal{O}_v/(t^2)) \rightarrow \text{sl}_2(\mathcal{O}_v/(t)) \rightarrow 0$$

From this we get

$$0 \rightarrow H^1(\text{Ad}_{\bar{\rho}}) \rightarrow H^1(\text{Ad}_{\rho'}) \rightarrow H^1(\text{Ad}_{\bar{\rho}}) \xrightarrow{B} H^2(\text{Ad}_{\bar{\rho}}) \rightarrow H^2(\text{Ad}_{\rho'}) \rightarrow H^2(\text{Ad}_{\bar{\rho}}) \rightarrow 0 \quad (5)$$

where $\text{Ad}_{\rho'}$ denotes the homology of the complex with coefficients in $\text{sl}_2(\mathcal{O}_v/(t^2))$. The computation of the connection operator proves that it is nothing but B we defined above. Write $H_1(M, \text{Ad}_{\rho})_v = \mathcal{O}_v \oplus \bigoplus_i \mathcal{O}_v/(t^{n_i})$, hence we have $H^1(M, \text{Ad}_{\bar{\rho}}) = \mathcal{O}_v/(t) \oplus \bigoplus_i \mathcal{O}_v/(t)$. Moreover we have $H_1(M, \text{Ad}_{\rho'}) \simeq \mathcal{O}_v/(t^2) \oplus \bigoplus_i \mathcal{O}_v/(t^{\min(n_i, 2)})$, and by the Universal Coefficients Theorem, $H^1(M, \text{Ad}_{\rho'}) \simeq \text{Hom}(H_1(M, \text{Ad}_{\rho'}), \mathcal{O}_v/(t^2)) \simeq H_1(M, \text{Ad}_{\rho'})$.

The first terms of equation (5) become

$$0 \rightarrow \mathcal{O}_v/(t) \oplus \bigoplus_i \mathcal{O}_v/(t) \xrightarrow{i} \mathcal{O}_v/(t^2) \oplus \bigoplus_i \mathcal{O}_v/(t^{\min(n_i, 2)}) \xrightarrow{p} \dots$$

The image of p is the kernel of B , thus it is a copy of $\mathcal{O}_v/(t)$ generated by v . Hence the image of is $\mathcal{O}_v/(t) \oplus \bigoplus_i \mathcal{O}_v/(t^{\min(n_i, 2)})$, and a simple count of dimensions proves that $n_i = 1$ for all i . In conclusion

$$H_1(M, \text{Ad}_{\rho})_v \simeq \mathcal{O}_v \oplus (\mathcal{O}_v/(t))^{2r-2}$$

and the torsion vanishes at order $2r - 2$ □

4 The torsion at an ideal point

In this section we consider $y \in Y$ an ideal point, let's recall that it is a point added "at infinity" by compactification. It corresponds to an unique valuation v on the field $k(Y)$, let's denote by t an uniforming parameter of the valuation ring \mathcal{O}_v .

We mentioned in Section 1.6 the construction of Culler and Shalen : such an ideal point induces an action of $\pi_1(M)$ on a simplicial tree T_v . In [Sha02] the autor explains how one can produce a so-called *dual surface* Σ to the action of $\pi_1(M)$ on T_v . Moreover, this surface can be chosen to be *incompressible* in the manifold M , that is such that the inclusion map $\pi_1(\Sigma) \rightarrow \pi_1(M)$ is injective. This section is devoted to the proof of the following theorem :

Theorem 4.1. *Assume that Σ is connected and splits M into two handlebodies M_1 and M_2 , such that the restricted tautological representation is not residually abelian on Σ . Then the following inequality holds :*

$$v(\text{tor}(M)) \leq -\chi(\Sigma) - 1$$

4.1 An overview of the construction of the dual surface Σ

Given a valuation v on the field $k(Y)$, the Bass-Serre tree T_v is a simplicial tree endowed with an action of $\pi_1(M)$. Given K a triangulation of M , and \tilde{K} its lift to the universal cover \tilde{M} of M , one can construct an equivariant simplicial map $f : \tilde{K} \rightarrow T_v$. If E is the set of midpoints of the edges of T_v , then by a transversality argument $S = f^{-1}(E)$ is an oriented, $\pi_1(M)$ -invariant surface, non empty as soon as the action of $\pi_1(M)$ does not fix a vertex of T_v , and we denote by Σ its projection in M . One can modify f such that the surface Σ is incompressible in M .

Define T as the *dual graph* of S in \tilde{M} , its vertices are in correspondence with the connected components of $\tilde{M} \setminus S$, and one join two vertices by an edge when a component of S lies between the two corresponding components of $\tilde{M} \setminus S$; moreover, one can assume $T \subset \tilde{K}$. Then f provides a simplicial embedding of T in T_v . Suppose that $M = M_1 \cap_\Sigma M_2$, then in T one can distinguish 2 classes of vertices, those that project onto M_1 or onto M_2 under the covering map $p : \tilde{M} \rightarrow M$. Take any edge e of T , its image in T_v has two endpoints s_1 and s_2 , one of each class described above. The equivariance of f implies that the action of $\pi_1(M_i)$ ($i = 1, 2$) through $\pi_1(M)$ stabilises s_i ($i = 1, 2$ respectively), and thus that the action of $\pi_1(\Sigma)$ stabilises e . We have proved the first part of the following lemma :

Lemma 4.2. *One can chose a conjugated of the tautological representation $\rho : \Gamma \rightarrow \text{SL}_2(k(Y))$ that restricts to representations ρ_1 and ρ_2 from $\pi_1(M_1), \pi_1(M_2)$ to $\text{SL}_2(k(Y))$ respectively ; such that ρ_1 is convergent and that ρ_Σ , its restriction to $\pi_1(\Sigma)$, is residually reducible.*

Proof. We just need to prove the last statement : again pick a basis such that s_1 corresponds to the lattice \mathcal{O}_v^2 , and that s_2 corresponds to $\mathcal{O}_v \oplus t\mathcal{O}_v$. As ρ_Σ fixes both s_1 and s_2 , then for all $\gamma \in \pi_1(\Sigma)$, $\rho_\Sigma(\gamma) = \begin{pmatrix} a(\gamma) & b(\gamma) \\ c(\gamma) & d(\gamma) \end{pmatrix}$, with $c(\gamma) \in (t)$, hence $\bar{\rho}_\Sigma$ is reducible. \square

The splitting $M = M_1 \cup_\Sigma M_2$ induces that one has the amalgamated product $\pi_1(M) = \pi_1(M_1) *_{\pi_1(\Sigma)} \pi_1(M_2)$. As ρ is not convergent, then it is not possible to chose it such that ρ_1 and ρ_2 take values simultaneously in $\text{SL}_2(\mathcal{O}_v)$. Again, if s_1 corresponds to \mathcal{O}_v^2 ,

then $\rho_1 : \pi_1(M_1) \rightarrow \mathrm{SL}_2(\mathcal{O}_v)$ does, and there is a matrix $A \in \mathrm{GL}_2(k(Y))$ such that $A\rho_2A^{-1} : \pi_1(M_2) \rightarrow \mathrm{SL}_2(\mathcal{O}_v)$ does too.

Lemma 4.3. *There is a convergent representation $\rho'_2 : \pi_1(M_2) \rightarrow \mathrm{SL}_2(\mathcal{O}_v)$ such that $\rho'_2 = U\rho_2U^{-1}$, where $U = \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$.*

Proof. Let $A' = t^N A$, with N big enough such that the entries of A' are in \mathcal{O}_v . As \mathcal{O}_v is a PID, one can find matrices $P, Q \in \mathrm{GL}_2(\mathcal{O}_v)$ such that $A' = P \begin{pmatrix} t^a & 0 \\ 0 & t^b \end{pmatrix} Q$. As the effect of conjugation doesn't change by multiplying M by a scalar, one can modify the matrix A such that $A' = P \begin{pmatrix} t^n & 0 \\ 0 & 1 \end{pmatrix} Q$, hence $\rho'_2 = P \begin{pmatrix} t^n & 0 \\ 0 & 1 \end{pmatrix} Q \rho_2 \left(P \begin{pmatrix} t^n & 0 \\ 0 & 1 \end{pmatrix} Q \right)^{-1}$, thus $P^{-1}\rho'_2P = \begin{pmatrix} t^n & 0 \\ 0 & 1 \end{pmatrix} Q \rho_2 Q^{-1} \begin{pmatrix} t^{-n} & 0 \\ 0 & 1 \end{pmatrix}$. But P, Q preserve convergence, hence we can change ρ to $Q\rho Q^{-1}$ and ρ'_2 to $P^{-1}\rho'_2P$. We conclude by noting that n is the distance in the tree T between the vertices s_1 and s_2 , so $n = 1$. \square

4.2 Proof of theorem 4.1

Consider now the splitting $M \setminus \Sigma = M_1 \cup M_2$, it induces the following Mayer-Vietoris exact sequence of $k(Y)$ -vector spaces (\mathcal{H}) :

$$0 \rightarrow H^1(M, \mathrm{Ad}_\rho) \xrightarrow{(i_1^*, i_2^*)} H^1(M_1, \mathrm{Ad}_{\rho_1}) \oplus H^1(M_2, \mathrm{Ad}_{\rho_2}) \xrightarrow{j_1^* - j_2^*} H^1(\Sigma, \mathrm{Ad}_\rho) \xrightarrow{\delta} H^2(M, \mathrm{Ad}_\rho) \rightarrow 0$$

Notice that the first term is zero because ρ_Σ is not abelian. Moreover, we can consider the complexes $C^*(\Sigma, \mathrm{Ad}_{\rho_\Sigma})$ and $C^*(M_i, \mathrm{Ad}_{\rho_i})$, $i = 1, 2$, with their natural geometric basis, and with a choice of basis h_Σ, h_1, h_2 of their homology groups. Here we need to take bases that generate the cohomology groups as \mathcal{O}_v -modules, that can be done because those modules are free, see Remark 4.5, item 3. That allows us to define their torsion, $\mathrm{tor}(\Sigma, h_\Sigma), \mathrm{tor}(M_i, h_i) \in k(Y)$ a priori. In fact, as the restricted representations are convergent, the torsion lies in the group \mathcal{O}_v^* .

We pick any basis of $H^1(M, \mathrm{Ad}_\rho)$, and again H as a basis of $H^2(M, \mathrm{Ad}_\rho)$. Then we have the following theorem of Milnor [Mil66] :

Theorem 4.4.

$$\mathrm{tor}(M) = \frac{\mathrm{tor}(M_1) \mathrm{tor}(M_2)}{\mathrm{tor}(\Sigma)} \mathrm{tor}(\mathcal{H}) \in \Omega_{k(Y)/k}$$

This equality (and the left-hand side term) does not depend on the choice of bases we made.

As $\mathrm{tor}(M_1), \mathrm{tor}(M_2)$ and $\mathrm{tor}(\Sigma)$ take values in \mathcal{O}_v^* , the valuation of $\mathrm{tor}(M)$ is determined by the torsion of the exact sequence \mathcal{H} .

Remark 4.5. 1. As we wish to compute the valuation of the torsion, it would be better to study a complex of \mathcal{O}_v -modules, having in mind Theorem 3.3. That will be the first step of the proof.

2. Notice that we have isomorphisms $k(Y) \xrightarrow{\sim} H^1(M, \mathrm{Ad}_\rho), 1 \mapsto (\frac{d}{dt}\rho)\rho^{-1}$ and $H^2(M, \mathrm{Ad}_\rho) \xrightarrow{\sim} k(Y), \lambda \mapsto \mathrm{Tr}(\lambda(\partial M)H)$.

3. As ρ_Σ is residually non abelian, so are ρ_1, ρ'_2 , and thus the only cohomology groups that are non trivial as \mathcal{O}_v -modules for them are $H^1(M_1, \text{Ad}_{\rho_1})_v$, $H^1(M_2, \text{Ad}_{\rho'_2})_v$ and $H^1(\Sigma, \text{Ad}_{\rho_\Sigma})_v$. It comes from Universal Coefficients Theorem that they are consequently free modules.

Lemma 4.6. *The sequence of $k(Y)$ -vector spaces*

$$0 \rightarrow k(Y) \xrightarrow{d_1} H^1(M_1, \text{Ad}_{\rho_1}) \oplus H^1(M_2, \text{Ad}_{\rho'_2}) \xrightarrow{d_2} H^1(\Sigma, \text{Ad}_{\rho_\Sigma}) \xrightarrow{\delta} k(Y) \rightarrow 0$$

is exact, where the morphisms are given by $d_1 : 1 \mapsto \left(\left(\frac{d}{dt} \rho_1 \right) \rho_1^{-1}, \left(\frac{d}{dt} \rho'_2 \right) \rho'^{-1}_2 \right)$, $d_2 : (\zeta_1, \zeta_2) \mapsto \zeta_1 - U^{-1} \zeta_2 U = \zeta_1 - \begin{pmatrix} x_2 & y_2/t \\ tz_2 & -x_2 \end{pmatrix}$ and $\delta : \lambda \mapsto \text{Tr}(\lambda([\partial\Sigma])H)$.

Proof. We just used the isomorphism $H^1(M_2, \text{Ad}_{\rho_2}) \xrightarrow{\sim} H^1(M_2, \text{Ad}_{\rho'_2})$ given by $\zeta_2 \mapsto U\zeta_2 U^{-1}$ and rewrite the sequence. Then we compute the morphisms d_1, d_2 and δ :

$$\begin{aligned} U \left(\frac{d}{dt} \rho_2 \right) \rho_2^{-1} U^{-1} &= U \frac{d}{dt} (U^{-1} \rho'_2 U) (U^{-1} \rho'^{-1}_2 U) U^{-1} \\ &= \left(\frac{d}{dt} \rho'_2 + U \frac{d}{dt} U^{-1} \rho'_2 + \rho'_2 \frac{dU}{dt} U^{-1} \right) \rho'^{-1}_2 \\ &= \left(\frac{d}{dt} \rho'_2 \right) \rho'^{-1}_2 + \frac{1}{t} (\rho'_2 N \rho'^{-1}_2 - N) \end{aligned}$$

Observe then that the last term on the right is the boundary of $N_0 = N - \frac{1}{2} \text{Id}$, and the first assertion follows. The second is clear.

For the third point, the naturality of the Mayer-Vietoris sequence and the exact sequence of a pair provide the following diagram.

$$\begin{array}{ccc} H^1(\Sigma, \text{Ad}_\rho) & \xrightarrow{\delta} & H^2(M, \text{Ad}_\rho) \\ \downarrow & & \downarrow \\ H^1(\partial\Sigma, \text{Ad}_\rho) & \longrightarrow & H^2(\partial M, \text{Ad}_\rho) \xrightarrow{\sim} H^0(\partial M, \text{Ad}_\rho)^* \end{array}$$

As the second vertical arrow is an isomorphism, it's enough to compute the composition $H^1(\Sigma, \text{Ad}_\rho) \rightarrow H^1(\partial\Sigma, \text{Ad}_\rho) \rightarrow H^2(\partial M, \text{Ad}_\rho) \rightarrow k(Y)$, which is simply $\lambda \mapsto \text{Tr}(\lambda([\partial\Sigma])H)$. \square

Now each term of the sequence can be thought as an \mathcal{O}_v -module tensorized by $k(Y)$, but the map d_2 does not restrict to a morphism of \mathcal{O}_v -module. Hence in the sequel we will consider the exact sequence \mathcal{H}_t :

$$0 \rightarrow k(Y) \xrightarrow{d_1} H^1(M_1, \text{Ad}_{\rho_1}) \oplus H^1(M_2, \text{Ad}_{\rho'_2}) \xrightarrow{t \cdot d_2} H^1(\Sigma, \text{Ad}_{\rho_\Sigma}) \xrightarrow{\delta} k(Y) \rightarrow 0$$

where we just have multiplied d_2 by t . We will denote by D_2 this new map, which restricts to morphism of \mathcal{O}_v -modules $H^1(M_1, \text{Ad}_{\rho_1})_v \oplus H^1(M_2, \text{Ad}_{\rho'_2})_v \xrightarrow{D_2} H^1(\Sigma, \text{Ad}_{\rho_\Sigma})_v$.

From now on we consider that the choices of bases we made h_Σ, h_1 and h_2 gave splittings $H^1(M_1, \text{Ad}_{\rho_1}) \oplus H^1(M_2, \text{Ad}_{\rho'_2}) = \ker d_2 \oplus E_1$, and $H^1(\Sigma, \text{Ad}_{\rho_\Sigma}) = d_2(E_1) \oplus E_2$. Let Δ_2 be the restricted map $D_{2|_{E_1}} : E_1 \rightarrow d_2(E_1)$.

Lemma 4.7.

$$\text{tor}(\mathcal{H}_t) = \frac{1}{\det \Delta_2} c, \text{ with } c \in \mathcal{O}_v^*$$

Proof. Considering the definition of the torsion of Section 2.1.2, the following equality holds :

$$\text{tor}(\mathcal{H}_t) = \frac{\det(d_1 : k(Y) \rightarrow d_1(k(Y))) \det(\delta : E_2 \rightarrow k(Y))}{\det D_2}$$

Then we conclude the proof noting that the numerator lies in \mathcal{O}_v^* . \square

Hence we are now reduced to compute $v(\det(D_2))$.

To do this, the idea is the following : recall that the completion of the valuation ring \mathcal{O}_v is isomorphic to $k[[t]]$, the ring of formal series. Consider a matrix $A \in \mathcal{M}_n(\mathcal{O}_v)$ as a formal series $A = \sum t^i A_i$, with $A_i \in \mathcal{M}_n(k)$, the problem is to compute the valuation of its determinant. If $\det A_0 \neq 0$, then A is invertible, $\det A \in \mathcal{O}_v^*$ and $v(\det A) = 0$. If not, we have $k^n \xrightarrow{A_0} k^n$ which is not invertible and define $H^0(A_0) = \ker A_0$, $H^1(A_0) = \text{coker } A_0$,

hence $H^0(A_0) \simeq H^1(A_0) \neq 0$. Pick $P, Q \in \text{GL}_n(k)$ such that $PA_0Q = \begin{pmatrix} 0 & 0 \\ 0 & I_{n-r_0} \end{pmatrix}$ is diagonal, where $r_0 = \dim \ker A_0$, and I_{n-r_0} is the $(n - r_0)$ identity matrix. Then to compute $\det A$, it's enough to compute the determinant of the $r_0 \times r_0$ first block of $A_1 + tA_2 \dots$. More precisely $\det A = t^{r_0} \det A'_1 + o(t^{r_0})$, where A'_1 is the restriction of $\sum t^i A_{i+1}$ to $H^0(A_0) \otimes k[[t]]$, followed by the projection $k[[t]]^n \rightarrow H^1(A_0) \otimes k[[t]]$.

One proceeds by induction, the argument is formalized in the following lemma :

Lemma 4.8. *Let $A : \mathcal{O}_v^n \rightarrow \mathcal{O}_v^n$ a morphism such that $\det A \neq 0$. Working in the completion $\hat{\mathcal{O}}_v$ if necessary, we define $A_{\geq 0} = A$, $A_{\geq i+1} = \frac{d}{dt} A_{\geq i}$ restricted to $H^0(A_{\geq i}(0)) \otimes k[[t]]$ followed by the projection $k[[t]]^{n-\sum_{k=0}^i r_k} \rightarrow H^1(A_{\geq i}(0)) \otimes k[[t]]$, and $r_i = \dim \ker A_{\geq i}(0)$. Then $\det(A) = t^{\sum r_i} c$, with $c \in k^*$.*

Proof. Define the sequence $(r_n)_n$ as in the lemma. As $\det A \neq 0$, there is an i_0 such that $r_{i_0} = 0$. Take $0 < i \leq i_0$, after fixing appropriated bases of $\ker A_{\geq i-1}(0)$, one write $A_{\geq i}(0)$ as a diagonal matrix, with r_{i-1} zeros on the diagonal, and 1's after. Then the classical formula for the determinant tells us that $\det A_{\geq i} = t^{r_i} \det A_{\geq i+1} + o(t^{r_i})$, and the result follows by induction. \square

We will apply this lemma to the morphism Δ_2 . Recall that for $\rho_1, \rho'_2, \rho_\Sigma : \Gamma \rightarrow \text{SL}_2(\mathcal{O}_v)$, we have the so-called residual representations $\bar{\rho}_1, \bar{\rho}_2, \bar{\rho}_\Sigma : \Gamma \rightarrow \text{SL}_2(k)$ taking values in the residual field k . Moreover, $\bar{\rho}_\Sigma = \bar{\rho}_{1|\Sigma} = \bar{U}^{-1} \bar{\rho}'_{2|\Sigma} \bar{U}$ is reducible, non abelian, thus we have as in Section 3.2.2 :

Lemma 4.9. *The residual representations have the form*

$$\bar{\rho}_{1,\Sigma} = \begin{pmatrix} \lambda & \lambda^{-1}u_1 \\ 0 & \lambda^{-1} \end{pmatrix}, \bar{\rho}_{2,\Sigma} = \begin{pmatrix} \lambda & 0 \\ \lambda u_2 & \lambda^{-1} \end{pmatrix}$$

with $\lambda \in H^1(\Sigma, k^*)$, $u_1 \in H^1(\Sigma, k_{\lambda^2})$, $u_2 \in H^1(\Sigma, k_{\lambda^{-2}})$ non trivial.

Moreover, let $v_i = (\frac{d}{dt} \rho_i)|_{t=0} \bar{\rho}_i^{-1}$, one has that the $v_i \in H^1(M_i, \text{Ad}_{\bar{\rho}_i})$, $i = 1, 2$ are non trivial, and $v_{1|\Sigma} = \begin{pmatrix} * & * \\ u_2 & * \end{pmatrix}$, $v_{2|\Sigma} = \begin{pmatrix} * & u_1 \\ * & * \end{pmatrix}$.

Proof. The expression of $\bar{\rho}_{2,\Sigma}$ follows from the conjugacy formula $\rho_1 = U^{-1}\rho'_2 U$ when restricted on $\pi_1(\Sigma)$, the u_i 's are non trivial because the residuals representations are not abelian.

Relations between v_i 's and u_i 's follow after deriving this formula $\rho_1 = U^{-1}\rho'_2 U$ at $t = 0$. \square

The former sequence becomes residually $\bar{\mathcal{H}}$:

$$0 \rightarrow k \xrightarrow{\bar{d}_1} H^1(M_1, \text{Ad}_{\bar{\rho}_1}) \oplus H^1(M_2, \text{Ad}_{\bar{\rho}_2}) \xrightarrow{\bar{D}_2} H^1(\Sigma, \text{Ad}_{\bar{\rho}_{1,\Sigma}}) \xrightarrow{\bar{\delta}} k \rightarrow 0$$

with $\bar{d}_1(1) = (v_1, v_2)$, and $\bar{D}_2(\zeta_1, \zeta_2) = \begin{pmatrix} 0 & y_{2,\Sigma} \\ 0 & 0 \end{pmatrix}$, where $y_{2,\Sigma}$ denotes the upper-right entry of ζ_2 , restricted to $\pi_1(\Sigma)$.

Again (see Section 3.2.2), the triangularity of the adjoint action of $\bar{\rho}_{i,\Sigma}$ provides the following splittings :

$$\begin{aligned} 0 \rightarrow K_1 \rightarrow \text{Ad}_{\bar{\rho}_{1,\Sigma}} &\rightarrow k_{\lambda^{-2}} \rightarrow 0 \\ 0 \rightarrow K_2 \rightarrow \text{Ad}_{\bar{\rho}_{2,\Sigma}} &\rightarrow k_{\lambda^2} \rightarrow 0 \\ 0 \rightarrow k_{\lambda^2} \rightarrow K_1 &\rightarrow k \rightarrow 0 \end{aligned}$$

and thus the exact sequences of k -vector spaces :

$$\begin{aligned} 0 \rightarrow H^1(\Sigma, K_2) \rightarrow H^1(\Sigma, \text{Ad}_{\bar{\rho}_{2,\Sigma}}) &\xrightarrow{p} H^1(\Sigma, k_{\lambda^2}) \rightarrow 0 \\ 0 \rightarrow H^0(\Sigma, k) \rightarrow H^1(\Sigma, k_{\lambda^2}) \rightarrow H^1(\Sigma, K_1) &\rightarrow H^1(\Sigma, k) \rightarrow 0 \end{aligned}$$

and

$$0 \rightarrow H^1(\Sigma, K_1) \rightarrow H^1(\Sigma, \text{Ad}_{\bar{\rho}_{1,\Sigma}}) \rightarrow \dots$$

We denote by j the composition $H^1(\Sigma, k_{\lambda^2}) \rightarrow H^1(\Sigma, K_1) \rightarrow H^1(\Sigma, \text{Ad}_{\bar{\rho}_{1,\Sigma}})$.

Lemma 4.10. *The space $\ker j$ is one dimensional, more precisely, it is generated by the image of $H^0(\Sigma, k)$ in $H^1(\Sigma, \text{Ad}_{\bar{\rho}_{1,\Sigma}})$, that is by $\partial_{1,\Sigma} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -2u_1 \\ 0 & 0 \end{pmatrix}$.*

Proof. We compute $\partial_{1,\Sigma} H = \bar{\rho}_{1,\Sigma} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \bar{\rho}_{1,\Sigma}^{-1} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and obtain the claimed result. \square

The inclusion $\Sigma \subset M_2$ induces $i : H^1(M_2, \text{Ad}_{\bar{\rho}_2}) \rightarrow H^1(\Sigma, \text{Ad}_{\bar{\rho}_{2,\Sigma}})$.

Lemma 4.11. $\dim \ker(j \circ p \circ i) : H^1(M_2, \text{Ad}_{\bar{\rho}_2}) \rightarrow H^1(\Sigma, \text{Ad}_{\bar{\rho}_{1,\Sigma}}) = -\chi(\Sigma)/2 + 1 + s$, with s an integer in $[0, -\chi(\Sigma) - 1]$.

Proof. Notice that $\dim H^1(\Sigma, \text{Ad}_{\bar{\rho}_\Sigma}) = -3\chi(\Sigma)$, $\dim H^1(\Sigma, k_{\lambda^2}) = -\chi(\Sigma)$ and $\dim H^1(M_i, \text{Ad}_{\bar{\rho}_i}) = -3/2\chi(\Sigma)$. As p is onto, $\ker p$ has dimension $\dim H^1(\Sigma, \text{Ad}_{\bar{\rho}_{2,\Sigma}}) - \dim H^1(\Sigma, k_{\lambda^2}) = -3\chi(\Sigma) - (-\chi(\Sigma)) = -2\chi(\Sigma)$. If i is injective and if $\ker p$ and $\text{im } i$ intersect transversally, then $\dim \ker p \cap \text{im } i = -\chi(\Sigma)/2$. We define the integer s by the formula $\dim \ker p \cap \text{im } i = -\chi(\Sigma)/2 + s$. Finally, by the preceding lemma $\ker j$ has dimension 1. Moreover, $p \circ i(v_2) = u_1 \neq 0 \in \ker j$ hence the dimension of $\ker(j \circ p \circ i) = -\chi(\Sigma)/2 + 1 + s$. If $s = -\chi(\Sigma)$, then $p \circ i = 0$, but $u_1 \neq 0$, hence the result. \square

Now we can give a proof of Theorem 4.1 :

Proof. First we compute r_0 , the dimension of the first homology group of $\bar{\mathcal{H}}$, i.e. $H^1(\bar{\mathcal{H}}) = \ker \bar{D}_2 / \text{im } \bar{d}_1$. From the preceding lemma, $\dim \ker \bar{D}_2 = -\chi(\Sigma)/2 + 1 + s + (-3/2\chi(\Sigma)) = -2\chi(\Sigma) + 1 + s$. Hence $r_0 = -2\chi(\Sigma) + s$. Let $r = \sum_{i \geq 1} r_i$, we have from Lemma 4.8 that $\det(D_2) = -2\chi(\Sigma) + s + r$, and $v(\text{tor}(\mathcal{H}_t)) = 2\chi(\Sigma) - s - r$. Finally, observe that $t^{\text{rk}(d_2)} \text{tor}(\mathcal{H}_t) = \text{tor}(\mathcal{H})$. As $\text{rk}(d_2) = -3\chi(\Sigma) - 1$, the theorem is proved. \square

Remark 4.12. In the proof we have denoted by s and r the integers that measure the defect of equality in the statement of Theorem 4.1. Notice that the hypothesis that those integers are 0 is generic, moreover as s is bounded by $-\chi(\Sigma) - 1$, one has the inequality $0 \leq v(\text{tor}(M)) + r$, hence it would be sufficient that r be 0 to state that the torsion does not have a pole at v .

4.3 Back to Examples

1. The trefoil knot.

The incompressible surface Σ is an annulus, hence $\rho_\Sigma : \mathbb{Z} \rightarrow \text{SL}_2(k(t))$ is abelian, and the theorem cannot apply. Nevertheless the torsion has a pole of order one at ideal points corresponding to Σ , hence the equality of theorem remains true.

2. The figure-eight knot.

There are two incompressible surfaces Σ_1 and Σ_2 that are two-holed tori, and the torsion vanishes at order 1 at each ideal point. Again the equality $1 = -\chi(\Sigma_i) - 1$ holds.

3. The knot 5.2.

There are two incompressible surfaces Σ_1 and Σ_2 , see Figure 1, and as explained in the introduction, again the equality of Theorem 4.1 holds.

4. The knot 6.1.

Again, there are two incompressible surface, the first of Euler characteristic -2 (a two-holed torus), and the second of Euler characteristic -6 (a two-holed genus 3 surface). At the corresponding ideal points, the vanishing order of $\text{tor}(M)$ is 1, respectively 5, that corresponds again with the equality case of the theorem.

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